Alan Teorisi

Gauss–Ostrogradski teoremi, Gauss yasası

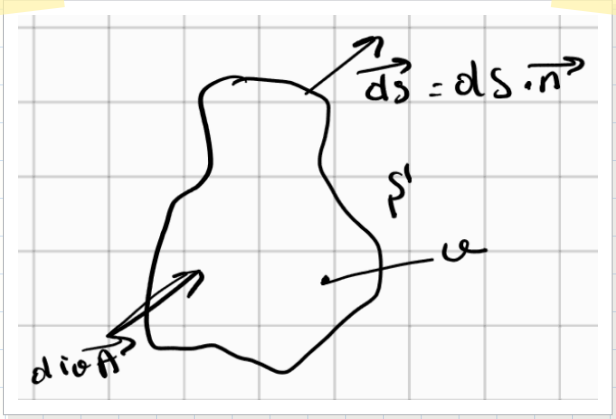
Orijinal not: Ders8.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

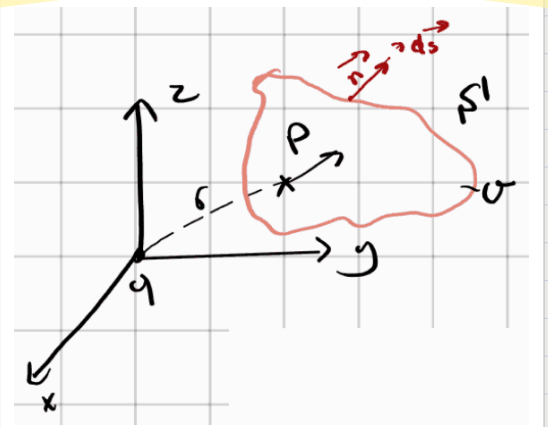
orj. s. 1

Gauss–Ostrogradski Teo:



orijinal çizim, s. 1

$$\int_V \operatorname{div} \vec{A} = \int_S \vec{A} \cdot ds$$



orijinal çizim, s. 1

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^2} = \frac{q\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

$$\vec{D} \triangleq \varepsilon_0\vec{E} = \frac{q}{4\pi}\frac{\vec{r}}{r^3}$$

↙ Deplasman
(yer değiştirme vektörü)

$$\int_S \vec{D}\cdot ds = \int_V \operatorname{div}\vec{D}\cdot dv = \int_V \operatorname{div}\left(\frac{q}{4\pi}\frac{\vec{r}}{r^3}\right) dv$$

$$= \frac{q}{4\pi}\int_V -\operatorname{div}\left(-\frac{\vec{r}}{r^3}\right) dv$$

$$= -\frac{q}{4\pi}\int_V \underbrace{\operatorname{div}\operatorname{grad}\left(\frac{1}{r}\right)}_{\Delta(1/r)} dv$$

$$\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z$$

$$f(x,y,z) \Rightarrow \operatorname{grad} f = \vec{\nabla}\cdot f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

orj. s. 2

$$\operatorname{div}\operatorname{grad} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f \qquad \underline{\text{laplacian}}$$

$$r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{1}{r} = (x^2+y^2+z^2)^{-1/2}$$

$$\Delta\left((x^2+y^2+z^2)^{-1/2}\right)$$

veya küresel koordinatlarda çözersiniz

$$f(r,\theta,\phi)^1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)}_{0}$$

$$+ \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}}_{0}$$

$$f = \frac{1}{r} = f(r)$$

$$\frac{\partial(1/r)}{\partial\theta} = \frac{\partial(1/r)}{\partial\phi} = 0$$

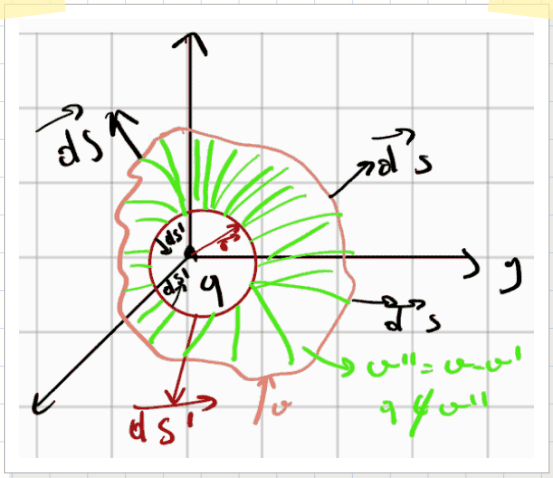
$$\Delta(1/r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\underbrace{\frac{\partial}{\partial r}(1/r)}_{-1/r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[-r^2\frac{1}{r^2}\right]$$

[1] Claude notu: muhtemelen $\Delta f(r,\theta,\phi)$ olmalı.

$$= -1 \frac{\partial}{\partial r}(1) = 0 \Rightarrow (r \neq 0)$$

$(r \neq 0)$ ise $q \notin v$ ise $\int_S \vec{D} \cdot ds = 0$

orj. s. 3



orijinal çizim, s. 3

S': r' yarıçaplı bir küre yüzeyi

v'' bölgesi $S \cup S'$ yüzeylerini kuşatır

$$\vec{r} \in v'' \Rightarrow \Delta(1/r) = 0$$

$$\Rightarrow \int_{v''} \operatorname{div} \vec{D} \cdot dv'' = \int_{S+S'} \vec{D}\,(d\vec{s} + d\vec{s}')$$

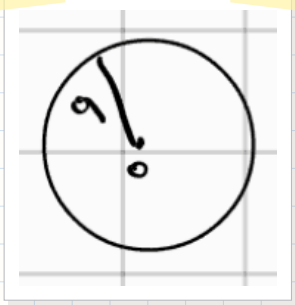
$$\Rightarrow -\frac{q}{4\pi} \int_{v''} \overbrace{\Delta(1/r)}^{0}\, dv''$$

$$= \int_{S+S'} \vec{D}\,(d\vec{s} + d\vec{s}') = 0 = \int_S \vec{D}\, d\vec{s} + \int_{S'} \vec{D} \cdot d\vec{s}' = 0$$

$$\int_S \vec{D} \cdot d\vec{s} = ?? \qquad (r = 0 \in v) \qquad q \in v$$

$$\int_S \vec{D} \cdot d\vec{s} = -\int_{S'} \vec{D} \cdot d\vec{s}'$$

q'nun S' de yarattığı $\vec{D}$: $\vec{D} = \frac{q}{4\pi} \frac{\vec{r}'}{(r')^3}$



orijinal çizim, s. 3

$$ds = a^2 \sin\theta \, d\theta \, d\phi$$
$$dv = r^2 \sin\theta \, d\theta \, d\phi^2$$

$$d\vec{s}' = -\vec{e}_{r'} \cdot \underbrace{r'^2 \sin\theta \, d\theta \, d\phi}_{ds'}$$

$$\vec{D} \cdot d\vec{s}' = \frac{q}{4\pi} \frac{r'\cdot}{r'^3} \left(\vec{e}_{r'} \cdot r'^2 \sin\theta' \, d\theta \, d\phi \right)$$

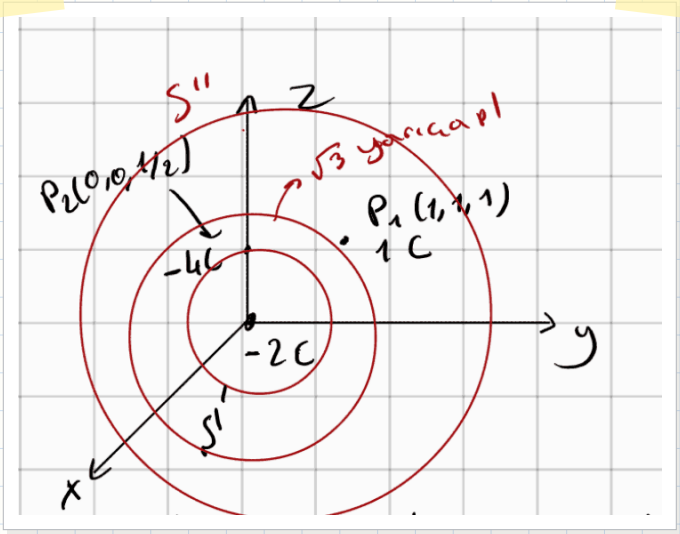
orj. s. 4

$$\int_{S'} \vec{D} \cdot ds' = -\int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} -\frac{q}{4\pi} \sin\theta' \, d\theta' \, d\phi'$$

$$= \frac{q}{4\pi} \underbrace{\int_{\theta'=0}^{\pi} \sin\theta' \, d\theta'}_{2} \underbrace{\int_{\phi'=0}^{2\pi} d\phi'}_{2\pi}$$

$$= \frac{q}{4\pi} \cdot 4\pi = q$$

$$\int_S \vec{D} \cdot d\vec{s} = Q_{iç} + \frac{Q_{yüzey}}{2}$$



orijinal çizim, s. 4

1/2 yarıçaplı küre yüzeyi
tekillik için koyulur yüzey üzerinde yük olduğu için

$$\int_S \vec{D} \cdot d\vec{s} = -2C + \left(\frac{-4C}{2} \right)$$
$$= -2C - 2C = -4C$$

$$[\vec{D}] = C/m^2$$

[2] Claude notu: muhtemelen $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$ olmalı.

$$[\vec{E}] = V/m \qquad [V] = V_o$$

$$\int_S \vec{D}\cdot d\vec{s} = -2C - 4C = -6C$$

$$\int_{S'} \vec{D}\cdot d\vec{s}' = -2C - 4C + 1C = -5C$$

orj. s. 5

Sınav 2 ya da 3 soru

$$\int_S \vec{D}\cdot d\vec{s} = \int_V \operatorname{div}\vec{D}\cdot dv = \underbrace{\int_V \rho\, dv}_{\text{v'deki Q yükü}}$$

$$= Q_{iç}$$

$$\rho = \frac{dQ}{dv} \qquad (\text{v içindeki h.y.y.} \quad [\rho] = C/m^3)$$

$$\boxed{\operatorname{div}\vec{D} = \rho} \leftarrow \textit{Maxwell}$$

Boşlukta

$$\vec{D} = \varepsilon_0\vec{E} \Rightarrow \operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{E} = -\operatorname{grad} V$$

$$\Rightarrow \underbrace{-\operatorname{div}\operatorname{grad} V}_{\Delta V} = \frac{\rho}{\varepsilon_0} \Rightarrow \Delta V = -\frac{\rho}{\varepsilon_0}$$

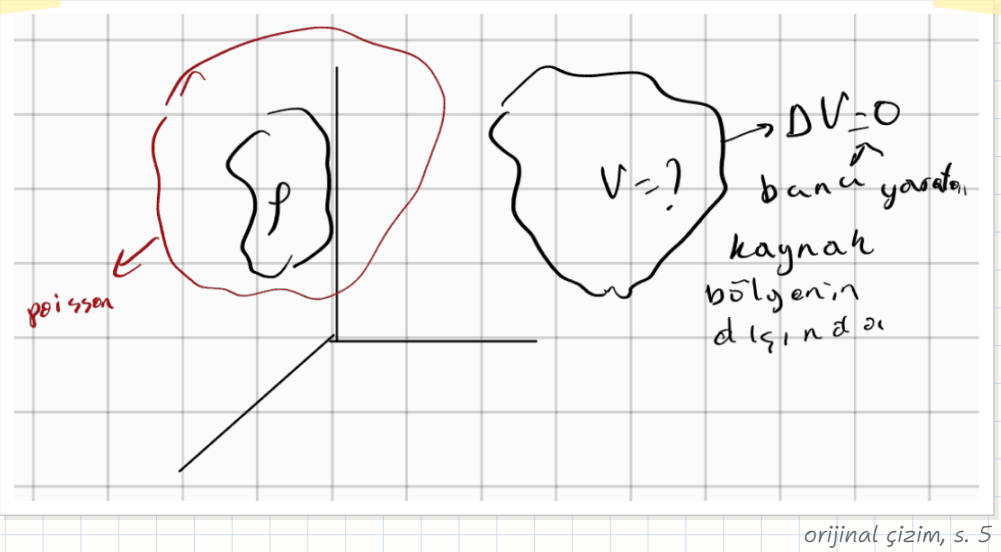
v'de $\rho \neq 0 \Rightarrow$ v'deki V pot fonksiyonun

$$\boxed{\Delta V = -\rho/\varepsilon_0} \Rightarrow V = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho}{r'}\, dv$$

↓ Poisson denklemi

v'de $\rho = 0$ (yani yük yok) $\Rightarrow$

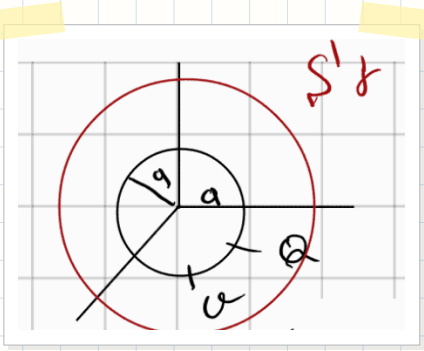
$$\boxed{\Delta V = 0} \leftarrow \textit{Laplace}$$



orijinal çizim, s. 5

orj. s. 6

S: 32
pr 5



orijinal çizim, s. 6

$$v = \frac{4\pi a^3}{3}$$

homojen yayılmış ↙

$$\boxed{\rho \triangleq \frac{dQ}{dv}} \Rightarrow \frac{Q}{v} = \rho = \frac{3Q}{4\pi a^3} \qquad (C/m^2)^3$$

$$Q = \int_v \rho\, dv = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{3Q}{4\pi a^3} r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{3Q}{4\pi a^3}\int_{r=0}^{a} r^2\, dr \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{3Q}{4\pi a^3}\,\frac{a^3}{3}\cdot 2\cdot 2\pi = Q$$

$$V = V(r) \quad , r > a \qquad \vec{E} = -\text{grad}\, V = -\frac{\partial V}{\partial r}\vec{e}_r$$

$$\vec{D} = \varepsilon_0 \vec{E} = -\underbrace{\varepsilon_0 \frac{\partial V}{\partial r}}_{D(r)} \vec{e}_r$$

$$d\vec{s} = r^2 \sin\theta\, d\theta\, d\phi\, \vec{e}_r$$

$$\int_{S_r} \vec{D}\cdot d\vec{s} = Q \Rightarrow \int_{S_r} \underbrace{D(r)\vec{e}_r}_{\vec{D}} \overset{!}{\cdot} \underbrace{\vec{e}_r \cdot r^2 \sin\theta\, d\theta\, d\phi}_{d\vec{s}_r} = Q$$

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D(r)\, r^2 \sin\theta\, d\theta\, d\phi = Q$$

orj. s. 7

$$\Rightarrow D(r)\, r^2 \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi = Q$$

$$D(r)\, r^2\, 4\pi = Q$$

$$D(r) = \frac{Q}{4\pi r^2} \Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\vec{e}_r$$

[3] Claude notu: hacimsel yük yoğunluğu için birim C/m^3 olmalı.

o zaman merkezde tek bir yük gibi küre davranır $r > a$ için

$$\vec{D}(r) = \frac{Q}{4\pi r^2}\vec{e}_r = -\varepsilon_0 \frac{\partial V}{\partial r}\vec{e}_r$$

$$V = \int \frac{\partial V}{\partial r}\,dr = \int \frac{-Q}{4\pi\varepsilon_0 r^2}\,dr \Rightarrow V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} + V_0$$

$$r \to \infty \Rightarrow V = 0 \Rightarrow V_0 = 0$$

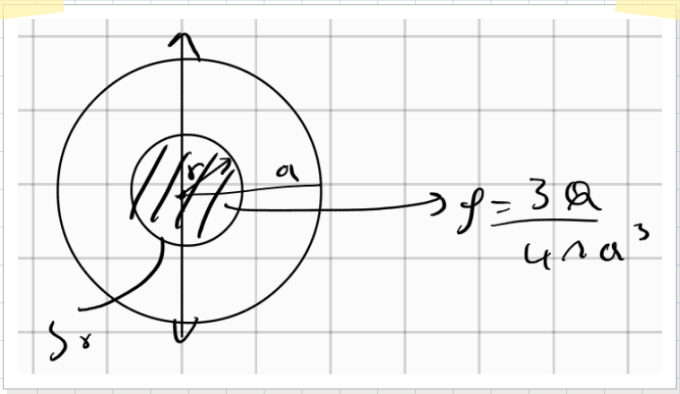
$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} + V_0$$

$$\lim_{r\to\infty} V(r) = V_0 = 0$$

$$r > a: \qquad V = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r}$$

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\underbrace{\Delta(1/r)}_{0} = 0$$

A $0 < r < a$



orijinal çizim, s. 7

orj. s. 8

$$\vec{D} = D(r)\vec{e}_r$$

$$\int_{S_r} \vec{D}\cdot d\vec{s}_r = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D(r)\underbrace{\vec{e}_r\cdot\vec{e}_r}_{1} r^2\sin\theta\,d\theta\,d\phi = \int_{v_r}\rho\,dv_r$$

$$D(r)\,r^2\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi = \int\frac{3Q}{4\pi a^3}\,dv_r$$

$$\Rightarrow D(r)\,r^2\,4\pi = \frac{3Q}{4\pi a^3}v_r = \frac{3Q}{4\pi a^3}\frac{4\pi r^3}{3}$$

$$\int_{v_r} dv_r = v_r = \frac{4\pi r^3}{3}$$

$$\int_{v_r} dv_r = \int_0^r\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\underbrace{r^2\sin\theta\,d\theta\,d\phi\,dr}_{dv_r = dr\,ds_r}$$

$$= \int_{r=0}^{r} r^2\, dr \underbrace{\int_{\theta=0}^{\pi} \sin\theta\, d\theta}_{2} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi}$$

$$\frac{r^3}{3}\Big|_0^r = \frac{r^3}{3}$$

$$\frac{4\pi r^3}{3}$$

$$\Rightarrow D(r)\, r^2\, 4\pi = \frac{Q \cdot r^3}{a^3}$$

$$\Rightarrow D(r) = \frac{Q\, r}{4\pi a^3}$$

$$\vec{D} = \boxed{\frac{Q}{4\pi a^3} r}\, \vec{e}_r = \boxed{-\varepsilon_0 \frac{\partial V}{\partial r}}\, \vec{e}_r$$

orj. s. 9

$$\frac{\partial V}{\partial r} = -\frac{Q}{4\pi\varepsilon_0 a^3} r$$

$$V = -\frac{Q}{8\pi\varepsilon_0 a^3} r^2 + \widetilde{V_0}$$

$$\underbrace{V\big|_{r=a^+}}_{\downarrow} = \underbrace{V\big|_{r=a^-}}_{\downarrow}$$

$$\frac{Q}{4\pi\varepsilon_0} \frac{1}{r}\Big|_{r=a} = -\frac{Q}{8\pi\varepsilon_0 a^3} r^2\Big|_{r=a} + \widetilde{V_0}$$

$$\frac{Q}{4\pi\varepsilon_0} \frac{1}{a} = -\frac{Q}{8\pi\varepsilon_0 a} + \widetilde{V_0}$$

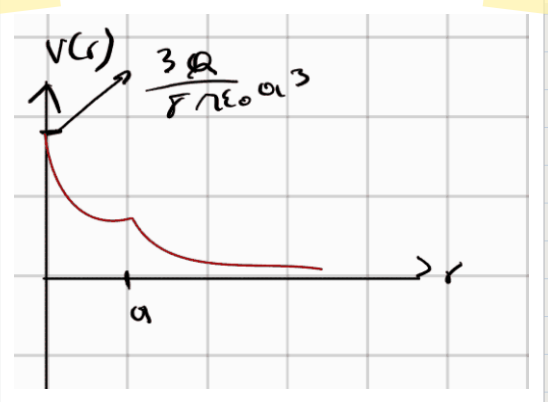
$$\widetilde{V_0} = \frac{3Q}{8\pi\varepsilon_0 a}$$

$0 < r < a$

$$\boxed{V(r) = -\frac{Q}{8\pi\varepsilon_0 a}\left[\frac{r^2}{a^2} - 3\right]}$$

$r > a$:

$$V(r) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r}$$



orijinal çizim, s. 9

$$V(r) = -\frac{Q r^2}{8\pi\varepsilon_0 a^3} + \frac{3Q}{8\pi\varepsilon_0 a}$$

$$\Delta V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\frac{Q}{8\pi\varepsilon_0 a^3}\,\Delta(r^2)$$

orj. s. 10

$$= -\frac{Q}{8\pi\varepsilon_0 a^3} \cdot \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left[\overbrace{r^2 \frac{\partial (r^2)}{\partial r}}^{2r^3}\right]$$

$$= -\frac{Q}{4\pi\varepsilon_0 a^3} \cdot \frac{3r^2}{r^2}[4] = -\boxed{\frac{3Q}{4\pi a^3}}\,\frac{\rho}{\varepsilon_0}$$

$$= -\frac{\rho}{\varepsilon_0}$$

[4] El yazısında pay ve paydaki r^2 terimleri çizilerek sadeleştirilmiş.