Alan Teorisi

Vektör analizi: grad, div, rot ve koordinat sistemleri

Orijinal not: Uygulama1.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1

Salı, 1-3

Uyg 1

$$\underset{\text{türev alıcı}}{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \quad : \text{Nabla operatörü}$$

Gradyan

$f(x, y, z)$ fonksiyonu olsun skaler (örn: $ax + by + cz$)

$$\underbrace{\nabla f(x,y,z)}_{* \,=\, \text{grad}\, f} = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial y}\vec{e}_z \text{[1]} \quad : \text{Gradyant vektöreldir}$$

skaler → [grad] → vektör

Divergans

$$\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z$$

$$\nabla \cdot \vec{A} = \operatorname{div}\vec{A} = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right)\left(A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z\right)$$

$$\operatorname{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

vektör → [div] → skaler

$\vec{e}_x \cdot \vec{e}_x = 1$ $\quad \vec{e}_x \times \vec{e}_x = 0$

$\vec{e}_x \cdot \vec{e}_y = 0$ $\quad \vec{e}_x \times \vec{e}_y = \vec{e}_z$

$\vec{e}_y \times \vec{e}_x = -\vec{e}_z$

$\vec{e}_y \times \vec{e}_z = \vec{e}_x$

$x \to y \to z$ (döngüsel, ok z'den x'e geri döner)

[1] Claude notu: muhtemelen son terimde $\partial f/\partial z$ olmalı.

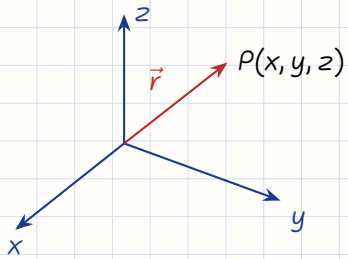
Rotasyonel

$$\underset{\text{rot}\,A}{\nabla \times \vec{A}} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

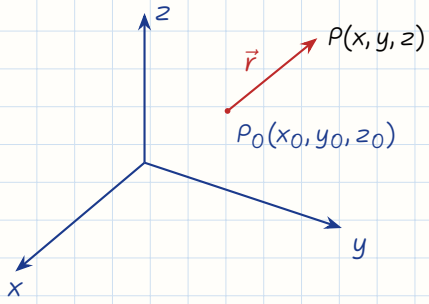
$$= \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\vec{e}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\vec{e}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\vec{e}_z$$

Vektör → [rot] → vektör

orj. s. 2



konum vektörü: $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$



$$\vec{r} = (x - x_0)\vec{e}_x + (y - y_0)\vec{e}_y + (z - z_0)\vec{e}_z$$

$$|\vec{r}| = r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$* \ \text{grad}\left(\frac{1}{r}\right) = ? = -\frac{\vec{r}}{r^3}$$

$$\text{grad}\left(\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}\right) = \frac{\partial}{\partial x}\frac{1}{r}\vec{e}_x + \frac{\partial}{\partial y}\frac{1}{r}\vec{e}_y + \frac{\partial}{\partial z}\frac{1}{r}\vec{e}_z$$

$$\frac{\partial}{\partial x}\frac{1}{r} = \frac{\partial}{\partial x}\underbrace{\left(\frac{1}{\underbrace{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}_{\upsilon}}\right)}_{f(\upsilon)} = \left(\frac{\partial f(\upsilon)}{\partial \upsilon}\right)\frac{\partial \upsilon}{\partial x}$$

$$f(\upsilon) = \frac{1}{\sqrt{\upsilon}}$$

$$\frac{\partial f(\upsilon)}{\partial \upsilon} = \frac{\partial}{\partial \upsilon}\left(\frac{1}{\sqrt{\upsilon}}\right) = -\frac{1}{2}\upsilon^{-3/2}$$

$$\frac{\partial \upsilon}{\partial x} = \frac{\partial}{\partial x}\left[(x - x_0)^2 + (y - y_0)^2 - (z - z_0)^2\right] = 2(x - x_0)^2$$

$$= -\frac{1}{2}\upsilon^{-3/2}\,2(x - x_0) = -\frac{x - x_0}{\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{3/2}}$$

[2] Claude notu: muhtemelen $+(z - z_0)^2$ olmalı.

$$\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y-y_0}{[(\cdots)]^{3/2}} \qquad \frac{\partial}{\partial z}\frac{1}{r} = -\frac{z-z_0}{[(\cdots)]^{3/2}}$$

orj. s. 3

$$\operatorname{grad}\left(\frac{1}{r}\right) = \underbrace{-\frac{1}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}}_{-1/r^3}\underbrace{[(x-x_0)\vec{e}_x+(y-y_0)\vec{e}_y+(z-z_0)\vec{e}_z]}_{\vec{r}}$$

$$\operatorname{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

2) $*\ \operatorname{rot}\operatorname{grad} f = 0$ Böyle olduğunu gösteriniz.

$f(x,y,z)$ skaler bir fonksiyon

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\operatorname{rot}\operatorname{grad} f = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$-\begin{matrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{matrix}$$

$$= \cancel{\frac{\partial^2 f}{\partial y\,\partial z}\vec{e}_x} + \cancel{\frac{\partial^2 f}{\partial x\,\partial y}\vec{e}_z} + \cancel{\frac{\partial^2 f}{\partial z\,\partial x}\vec{e}_y}$$

$$+ \left(-\cancel{\frac{\partial^2 f}{\partial y\,\partial x}\vec{e}_z} - \cancel{\frac{\partial^2 f}{\partial z\,\partial y}\vec{e}_x} - \cancel{\frac{\partial^2 f}{\partial x\,\partial z}\vec{e}_y}\right)$$

$$= 0$$

3) $\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z$
$\operatorname{div}\operatorname{rot}\vec{A} = 0$ olduğunu gösteriniz.

$$\operatorname{rot}\vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\vec{e}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\vec{e}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\vec{e}_z$$

$$\operatorname{div}\operatorname{rot}\vec{A} = \frac{\partial}{\partial x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$= \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z} + \frac{\partial^2 A_x}{\partial y\,\partial z} - \frac{\partial^2 A_z}{\partial y\,\partial x} + \frac{\partial^2 A_y}{\partial z\,\partial x} - \frac{\partial^2 A_x}{\partial z\,\partial y}$$

orj. s. 4

$$\operatorname{div}\operatorname{rot}\vec{A} = 0$$

4) $f(x,y,z)$ skaler fonksiyon $\vec{A}(x,y,z)$ vektör

$\operatorname{div}(f\cdot\vec{A}) = f\operatorname{div}\vec{A} + \vec{A}\cdot\operatorname{grad} f$ olduğunu gösteriniz.

$$\operatorname{div}(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= \left[f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x}\right] + \left[f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y}\right] + \left[f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial z}\right]$$

$$= f\underbrace{\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]}_{\operatorname{div}\vec{A}} + A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}$$

$$\left\{\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z\right\}$$

$$= f\operatorname{div}\vec{A} + \vec{A}\cdot\operatorname{grad} f$$

$*5)$

$$\operatorname{rot}(f\vec{A}) = f\operatorname{rot}\vec{A} - \vec{A}\times\operatorname{grad} f \quad \text{(cross product)}$$

$$\left\{f\vec{A} = fA_x\vec{e}_x + fA_y\vec{e}_y + fA_z\vec{e}_z\right\}$$

$$= \left[\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right]\vec{e}_x + \left[\frac{\partial}{\partial z}(fA_x) - \frac{\partial}{\partial x}(fA_z)\right]\vec{e}_y + \left[\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right]\vec{e}_z$$

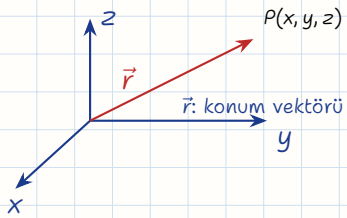
$$\operatorname{rot}(f\vec{A}) = \left[f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right]\vec{e}_x + \left[f\frac{\partial A_x}{\partial z} + A_x\frac{\partial f}{\partial z} - f\frac{\partial A_z}{\partial x} - A_z\frac{\partial f}{\partial x}\right]\vec{e}_y + \left[f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right]\vec{e}_z$$

orj. s. 5

$$\operatorname{rot}(f\vec{A}) = f\overbrace{\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{e}_z\right]}^{\operatorname{rot}\vec{A}}$$
$$+ \underbrace{\left[\left(A_z\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial z}\right)\vec{e}_x + \left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial x}\right)^{3}\vec{e}_y + \left(A_y\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial y}\right)\vec{e}_z\right]}_{-\vec{A}\times\operatorname{grad} f}$$

$$= f\operatorname{rot}\vec{A} - \vec{A}\times\operatorname{grad} f$$

Kartezyan koordinatlar



$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

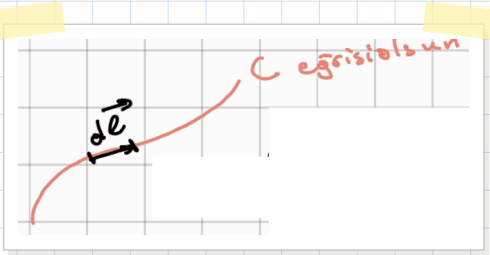
$$h = \begin{bmatrix} h_x & 0 & 0 \\ 0 & h_y & 0 \\ 0 & 0 & h_z \end{bmatrix} : \text{metrik tensörü}$$

$$\frac{\partial\vec{r}}{\partial x} = h_x\vec{e}_x = 1\cdot\vec{e}_x \Rightarrow h_x = 1$$

$$\frac{\partial\vec{r}}{\partial y} = h_y\vec{e}_y \qquad h_y = 1$$

$$\frac{\partial\vec{r}}{\partial z} = h_z\vec{e}_z \qquad h_z = 1$$

[3] Claude notu: muhtemelen $(A_x\,\partial f/\partial z - A_z\,\partial f/\partial x)$ olmalı.



orijinal çizim, s. 5

$$\xrightarrow[\text{uzunluk}] \quad d\vec{\ell} = h_x\,dx\,\vec{e}_x + h_y\,dy\,\vec{e}_y + h_z\,dz\,\vec{e}_z$$

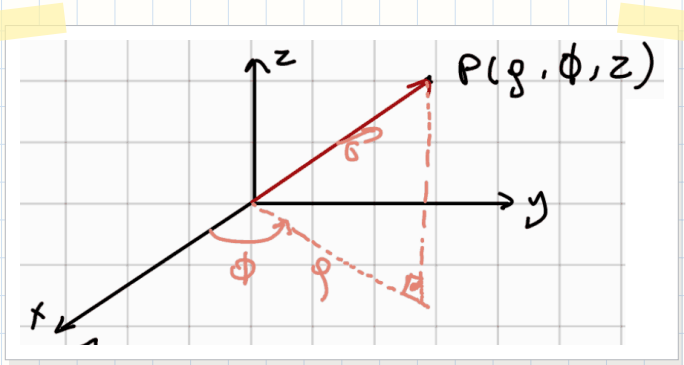
$$= dx\,\vec{e}_x + dy\,\vec{e}_y + dz\,\vec{e}_z$$

$$\xrightarrow{\text{hacim}} \quad dV = h_x h_y h_z\,dx\,dy\,dz$$

$$= dx\,dy\,dz$$

orj. s. 6

Silindirik koordinatlar



orijinal çizim, s. 6

$$\begin{bmatrix} h_\rho & & \\ & h_\phi & \\ & & h_z \end{bmatrix}$$

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$\vec{r} = \rho\cos\phi\,\vec{e}_x + \rho\sin\phi\,\vec{e}_y + z\,\vec{e}_z$$

$$\frac{\partial\vec{r}}{\partial\rho} = \cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y = h_\rho\vec{e}_\rho$$

$$h_\rho = 1, \qquad \vec{e}_\rho = \cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y$$

$$\frac{\partial\vec{r}}{\partial\phi} = -\rho\sin\phi\,\vec{e}_x + \rho\cos\phi\,\vec{e}_y = h_\phi\vec{e}_\phi$$

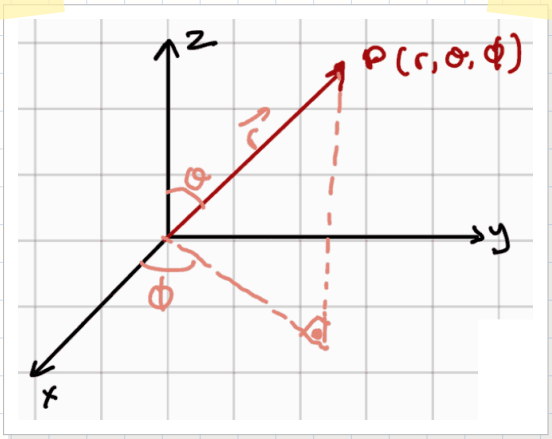
$$h_\phi = \rho, \qquad \vec{e}_\phi = -\sin\phi\,\vec{e}_x + \cos\phi\,\vec{e}_y$$

$$* \quad d\vec{\ell} = h_\rho\,d\rho\,\vec{e}_\rho + h_\phi\,d\phi\,\vec{e}_\phi + h_z\,dz\,\vec{e}_z$$

$$= d\rho\,\vec{e}_\rho + \rho\,d\phi\,\vec{e}_\phi + dz\,\vec{e}_z$$

$$dV = \rho\,d\rho\,d\phi\,dz$$

Küresel koordinatlar



orijinal çizim, s. 6

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\vec{r} = r\sin\theta\cos\phi\,\vec{e}_x + r\sin\theta\sin\phi\,\vec{e}_y + r\cos\theta\,\vec{e}_z$$

$h_r = 1,$ $\quad \vec{e}_r = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z$

$h_\theta = r,$ $\quad \vec{e}_\theta = \cos\theta\cos\phi\,\vec{e}_x + \cos\theta\sin\phi\,\vec{e}_y - \sin\theta\,\vec{e}_z$

$h_\phi = r\sin\theta,$ $\quad \vec{e}_\phi = -\sin\phi\,\vec{e}_x + \cos\theta\,\vec{e}_y$[4]

$$d\vec{\ell} = dr\,\vec{e}_r + r\,d\theta\,\vec{e}_\theta + r\sin\theta\,d\phi\,\vec{e}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

orj. s. 7

$\phi : 0 \to 2\pi$
$\theta : 0 \to \pi$

[4] Claude notu: muhtemelen cos φ olmalı.