Analog Haberleşme

Açı modülasyonu, Carson kuralı, tek tonlu FM/PM

Orijinal not: Ders12.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1

Açı modülasyonu

$$y(t) = A_c \cos\left(\overbrace{2\pi f_c t + \phi(t)}^{\theta(t)}\right)$$

$$\hookrightarrow k_p\, m(t) \quad \text{FZM} \quad \text{PM}$$

$$= 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau \quad \text{FRM}$$

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

$$\theta(t) = 2\pi \int_{-\infty}^{t} f_i(\tau)\, d\tau$$

$$P_y = \frac{A_c^2}{2}$$

$$BG = \infty \quad \text{(Teorik)}$$

Carson kuralı

(Açı modülasyonunun efektif BG)

$$BG\{m(t)\} = B \text{ Hz}$$

$$BG = 2B(\beta + 1) \text{ Hz}, \qquad \underset{\text{sapma oranı}}{\beta} = \frac{\Delta f}{B}$$

$$\text{FM(FRM)} \quad f_i(t) = f_c + k_f m(t) \rightarrow \Delta f = k_f m_p, \quad m_p = \max|m(t)|$$

$$\text{PM(FZM)} \quad f_i(t) = f_c + \frac{k_p}{2\pi}\dot{m}(t)$$

$$\rightarrow \Delta f = \frac{k_p}{2\pi} m_p', \quad m_p' = \max|\dot{m}(t)|$$

[$\dot{m}(t)$ terimi altında küçük not: ? t=rev]

$$\Delta\phi = \max|\phi(t)|$$

$$k_f, k_p \ll \rightarrow \Delta f \ll \rightarrow \beta \ll 1 \Rightarrow BG = 2B \quad \text{NBFM / NBPM}$$

$$k_f, k_p \gg \rightarrow \Delta f \gg \rightarrow \beta \gg 1 \Rightarrow BG \approx 2\Delta f \quad \text{WBFM / WBPM}$$

Örnek

$m(t) = a\cos(2\pi f_m t)$
$c(t) = A_c\cos(2\pi f_c t)$

PM:

$$\phi(t) = k_p\, m(t) = \overset{\Delta f/B}{\underset{\beta}{k_p\, a}}\cos(2\pi f_m t) = \beta\cos(2\pi f_m t)$$

FM:

$$\phi(t) = 2\pi k_f \int m(\tau)\,d\tau = 2\pi k_f \int a\cos(2\pi f_m \tau)\,d\tau$$

$$= 2\pi k_f \frac{a\sin(2\pi f_m t)}{2\pi f_m} = \overset{k_f m_p}{\underset{\beta}{\frac{k_f a}{f_m}}}\sin(2\pi f_m t)$$

orj. s. 2

$$= \beta\sin(2\pi f_m t)$$

$$y(t) = \begin{cases} A_c\cos\left(2\pi f_c t + \beta\cos(2\pi f_m t)\right) & \text{PM} \\ A_c\cos\left(2\pi f_c t + \beta\sin(2\pi f_m t)\right) & \text{FM} \end{cases}$$

Saf sinüsoidal işaretin açı modülasyonu

$$m(t) = a\cos(2\pi f_m t) \Rightarrow y_{FM}(t) = A_c\cos\left(2\pi f_c t + \beta\sin(2\pi f_m t)\right),$$
$$\beta = \frac{k_f \cdot a}{f_m}$$
$$m(t) = a\sin(2\pi f_m t) \Rightarrow y_{PM}(t) = A_c\cos\left(2\pi f_c t + \beta\sin(2\pi f_m t)\right),$$
$$\beta = k_p \cdot a$$

$$\hat{y}(t) = A_c\, e^{j(2\pi f_c t + \beta\sin(2\pi f_m t))} = A_c\, e^{j2\pi f_c t}\cdot \underbrace{e^{j\beta\sin(2\pi f_m t)}}_{x(t)}$$

$$y(t) = Re\{\hat{y}(t)\}$$

$$x(t) = \underbrace{e^{j\beta\sin(2\pi f_m t)}}_{T_m = \frac{1}{f_m}\text{ ile periyodiktir}} = \sum_{n=-\infty}^{\infty} x_n\, e^{j2\pi n f_m t}$$

$$x_n = \frac{1}{T_m}\int_{T_m} x(t)\, e^{-j2\pi n f_m t}\,dt$$
$$= \frac{1}{T_m}\int_0^{T_m} e^{j\beta\sin(2\pi f_m t)}\cdot e^{-j2\pi n f_m t}\,dt$$

$$2\pi f_m t = \alpha \Rightarrow dt = \frac{d\alpha}{2\pi f_m}$$

$$x_n = \frac{1}{T_m}\int_0^{2\pi} e^{j\beta\sin(\alpha)} \cdot e^{-jn\alpha}\,\frac{d\alpha}{2\pi f_m}$$

[turuncu çizgilerle T_m ve f_m terimleri sadeleştirilmiş]

$$\Rightarrow x_n = \frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta\sin\alpha - n\alpha)}\,d\alpha$$

$$x_n = \frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta\sin\alpha - n\alpha)}\,d\alpha = J_n(\beta)$$

analitik olarak hesaplayamayız, numerik olarak hesaplanır

Bessel function of the 1st kind and nth order
(n. mertebeden 1. tür Bessel fonk.)

$$x(t) = \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{j2\pi n f_m t}$$

$$\hat{y}(t) = A_c\, e^{j2\pi f_c t} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{j2\pi n f_m t}$$

$$\hat{y}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{j[2\pi(f_c + nf_m)t]}$$

orj. s. 3

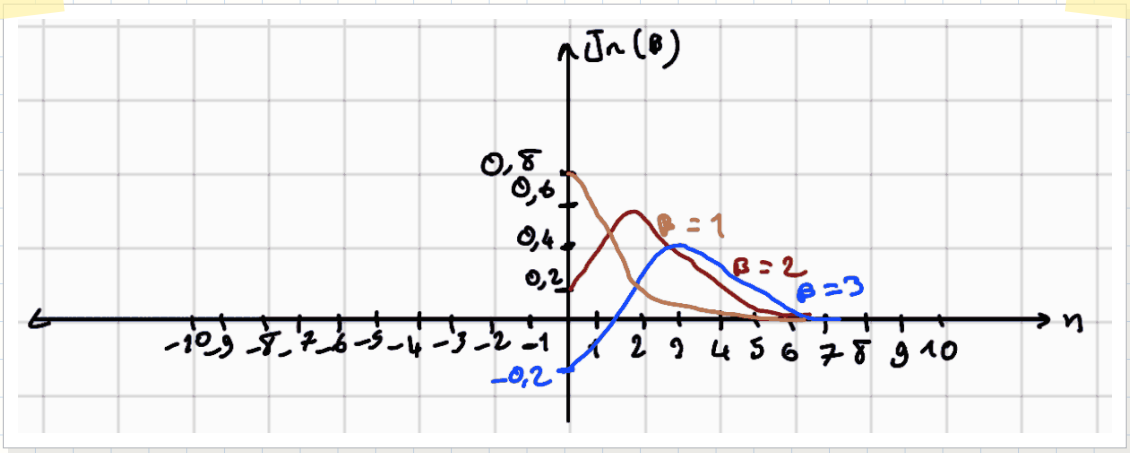
$$y(t) = \mathrm{Re}\{\hat{y}(t)\} = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cdot\cos\left(2\pi(f_c + nf_m)t\right)$$

$$y(t) = A_c\cos\left(2\pi f_c t + \beta\sin(2\pi f_m t)\right)$$
$$= A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos\left(2\pi(f_c + nf_m)t\right)$$

Bessel fonksiyonunu inceleyelim

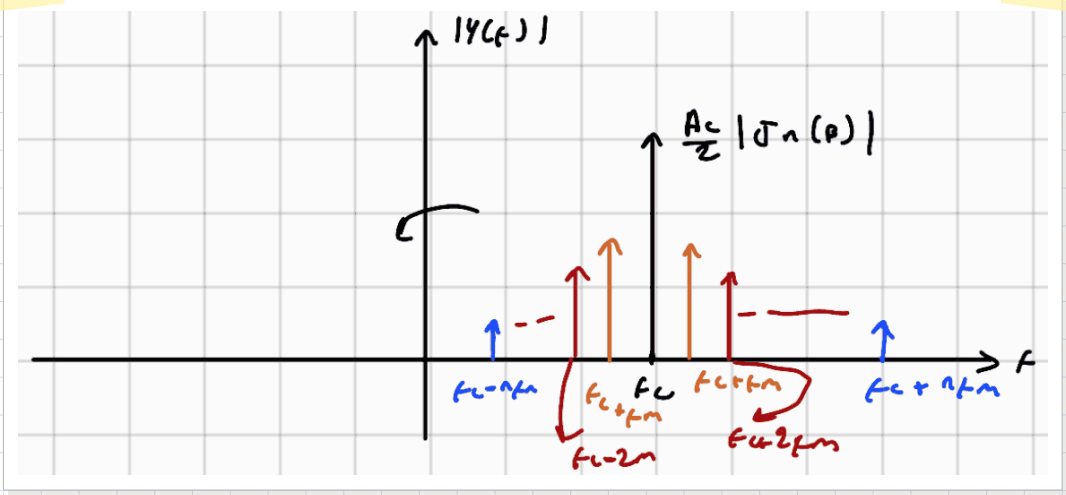
$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$
$$|J_{-n}(\beta)| = |J_n(\beta)|$$

Lathi (Proakis, s.176, Fig 4.4 benzeri)
↪ şekil oradan

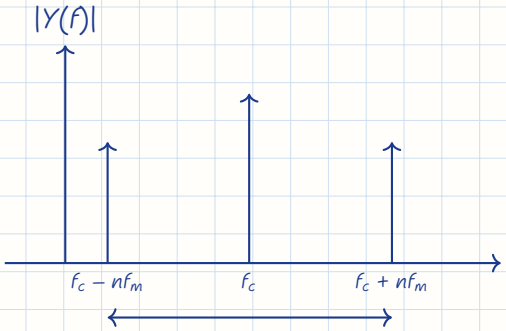


orijinal çizim, s. 3

$n > \beta + 1$ için $J_n(\beta) \ll$



orijinal çizim, s. 3



$$BG = 2nf_m$$
$$= 2(\beta + 1)f_m, \quad BG\{m(t)\} = f_m = B$$
$$= 2(\beta + 1)B = 2B(\beta + 1)$$

orj. s. 4

Açı modülasyonu lineer değildir.

Örnek

(Proakis)
$c(t) = 10\cos(2\pi f_c t)$, $m(t) = \cos(20\pi t)$
FM with $k_f = 50$
Find the expression for the modulated signal and determine how many harmonics should be selected to contain 99 % of the modulated signal power.

$$\phi(t) = 2\pi k_f \int m(\tau)\,d\tau = \frac{2\pi k_f}{20\pi}\sin(20\pi t)$$
$$= \underset{\beta}{5}\sin(20\pi t)$$

$$y(t) = A_c\cos\left(2\pi f_c t + \phi(t)\right)$$
$$y(t) = 10\cos\left(2\pi f_c t + \underbrace{5\sin(2\pi f_m t)}_{\phi(t)}\right)$$

$$A_c = 10, \quad \beta = \frac{\Delta f}{B} = \frac{\overset{50}{k_f}\cdot\overset{1}{m_p}}{10} = 5$$

$$y(t) = 10\cos\left(\underbrace{2\pi f_c t + 5\sin(20\pi t)}_{\theta(t)}\right)$$

$$\rightarrow\ P_y = \frac{A_c^2}{2} = 50 \text{ watt } (\%100)$$

$$y(t) = A_c\sum_{n=-\infty}^{+\infty} J_n(\beta)\cos\left(2\pi(f_c + nf_m)t\right)$$
$$= 10\sum_{n=-\infty}^{+\infty} J_n(5)\cos\left(2\pi(f_c + n\cdot 10)t\right)$$

$$P_y = \underset{50\text{ W }(\%100)}{\frac{A_c^2}{2}}\sum_{n=-\infty}^{+\infty} J_n^2(\beta)$$

$$\hookrightarrow \sum_{n=-\infty}^{+\infty} J_n^2(\beta) = 1$$

$$\Rightarrow \underbrace{\sum_{n=-k}^{k}}_{k \text{ tane harmonik}} J_n^2(\beta) \geq 0{,}99$$

$$J_0^2(5) + 2\sum_{n=1}^{k} J_n^2(5) \geq 0{,}99$$

Sayfa 176, Tablo 4.1'e göre

bu şart $k = 6$ için sağlanıyor
99% of the power is within
the frequencies $f_c \pm k\cdot 10 \rightarrow f_m$

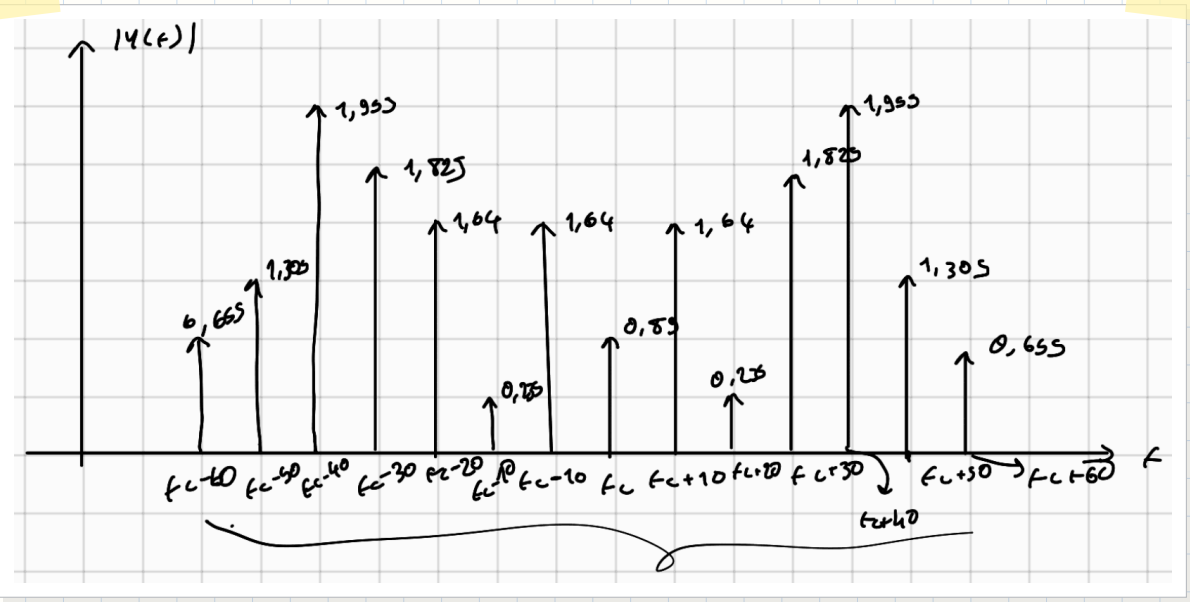
$\Big\}$ $f_c - 6\cdot 10$ ——— $f_c + 6\cdot 10$

B gerektiği $= 2 \times 6 \times 10 = 120$ Hz

orj. s. 5

$$\frac{2(\beta+1)\cdot f_m}{2(\beta+1)B\ \checkmark}$$

n	$J_n(5)$	$\frac{A_c}{2} J_n(5)$ $\left(\frac{A_c}{2}=5\right)$
0	−0,178	⟶ −0,89
1	−0,328	⟶ −1,64
2	0,047	⟶ 0,235
3	0,365	⟶ 1,825
4	0,391	⟶ 1,955
5	0,261	⟶ 1,305
6	0,131	⟶ 0,655



orijinal çizim, s. 5

$$BG = 120\ \text{Hz}$$

Mesajı gönderirken Band pass filtreden geçirip yollanabilir.
Bu bant genişliğinde bir IF kullanılabilir.

Örnek

Lathi, Ör. 5.2.6
$m(t) = \sin(2000\pi t)$
$k_f = 100\,000$, $k_p = 10$
a) Estimate BWs of $y_{FM}(t)$ and $y_{PM}(t)$
b) Repeat part (a) if the message signal amplitude is doubled
c) " frequency is "

orj. s. 6

d) Comment on the sensitivity of FM and PM BWs to the spectrum *(amplitude)* of $m(t)$

$$m(t) = \sin\left(2\pi \underset{B}{1000}\, t\right), \quad B = 1000\ \text{Hz}$$

a) FM: $\Delta f = k_f \cdot m_p \rightarrow \Delta f = 100$ kHz

$m_p = \max|m(t)| = 1$

$\downarrow \beta = \frac{\Delta f}{B} = 100$

$\hookrightarrow BG = 2B(\beta + 1) = 202$ kHz

PM: $\Delta f = \frac{k_p}{2\pi} \cdot m_p'$

$m_p' = \max|\dot{m}(t)| = \max|2\pi \cdot 1000 \cos(\ )| = 2000\pi$

$\Delta f = \frac{10}{2\pi} \times 2000\pi = 10$ kHz

$\rightarrow \beta = \frac{\Delta f}{B} = \frac{10 \text{ kHz}}{1 \text{ kHz}} = 10$

$BG_{PM} = 2B(\beta + 1) = 22$ kHz

b) $m(t) = 2\sin(2\pi \cdot 1000\, t)$

FM $m_p = 2 \rightarrow \Delta f = k_f m_p = 200$ kHz

$\rightarrow \beta = \frac{\Delta f}{B} = \frac{200 \text{ kHz}}{1 \text{ kHz}} = 200$

$\rightarrow BG_{FM} = 2B(\beta + 1) = 402$ kHz

PM $m_p' = 4000\pi \rightarrow \Delta f = \frac{k_p}{2\pi} \cdot m_p' = \frac{10}{2\pi} 4000\pi$

$= 20$ kHz

$\rightarrow \beta = \frac{\Delta f}{B} = 20 \Rightarrow BG_{PM} = 2B(\beta + 1) = 42$ kHz

d) iki modülasyonun da bant genişliği genliğe çok duyarlıdır

c)

$m(t) = \sin\left(2\pi \cdot \underset{B}{2000}\, t\right) \Rightarrow B = 2$ kHz

FM: $m_p = 1 \rightarrow \Delta f = k_f m_p = 100$ kHz

$\Rightarrow \beta = \frac{\Delta f}{B} = \frac{100k}{2k} = 50$

$\Rightarrow BG_{FM} = 2\underset{2k}{B}\left(\underset{50}{\beta} + 1\right) = 204$ kHz

orj. s. 7

PM: $m_p' = \max|\dot{m}(t)| = \max|2\pi\, 2000 \cos(2\pi\, 2000\, t)| = 4000\pi$

$\rightarrow \Delta f = \frac{k_p}{2\pi} \cdot m_p' = \frac{10}{2\pi} \cdot 4000\pi = 20$ kHz

$\rightarrow \beta = \frac{\Delta f}{B} = \frac{20k}{2k} = 10 \rightarrow BG_{PM} = 2B(\beta + 1)$

$= 2 \cdot 2k \times (10 + 1) = 44$ kHz

d) frekans modülasyonu frekansa çok duyarlı değil ama PM modülasyonu çok duyarlıdır.

Örnek

Lathi Ex 5.9

An angle modulated signal with carrier frequency $f_c = 10^6$ Hz is described by the equation

$$y_{EM}(t) = 10\cos\left[2\pi 10^6 t + 5\sin(3000t) + 10\sin(2000\pi t)\right]$$

[Turuncu/kalem notları: y_{EM}: exponential modulation (angle modulation); 10: A_c; $2\pi 10^6$: f_c; $5\sin(3000t) + 10\sin(2000\pi t)$: $\phi(t)$; tüm argüman: $\theta(t)$; $f_1 = \frac{3000}{2\pi}$; $f_2 = \frac{2000\pi}{2\pi} = 1\,\text{kHz}$; $BG\{m(t)\} = 1\,\text{kHz}$]

a) Find the power of the modulated signal

b) Find the max frequency deviation Δf

c) " the deviation ratio β

d) " the max phase deviation $\Delta\phi$

e) Estimate the BW of $y_{EM}(t)$

$$\Delta f = \max|f_i(t) - f_c|$$

a) $P_y = \frac{A_c^2}{2} = 50$ watt

b) $f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$

$$\Rightarrow \Delta f = \frac{1}{2\pi}\max|\dot{\phi}(t)|$$

$$\Delta f = \frac{1}{2\pi}\max\Big|5\times 3000\underbrace{\cos(3000t)}_{\approx 1} + 10\times 2000\pi\underbrace{\cos(2000\pi t)}_{\approx 1}\Big|$$

$$\Delta f \approx \frac{1}{2\pi}\left[15000 + 20000\pi\right] = 12388\ \text{Hz}$$

c) $\beta = \frac{\Delta f}{B} = \frac{12388}{1\,\text{kHz}}$

orj. s. 8

$$\phi(t) = \begin{cases} k_p\, m(t) \\ \text{veya} \\ 2\pi k_f \int m(\tau)\,d\tau \end{cases}$$

$$= \underbrace{5\sin(3000t)}_{f_1 = \frac{3000}{2\pi} \approx 477{,}5\ \text{Hz}} + \underbrace{10\sin(2000\pi t)}_{f_2 = 1000}$$

$$\Rightarrow BG = f_2 = 1\ \text{kHz}$$

d) $\Delta\phi = \max|\phi(t)|$

$$= \max\Big|5\underbrace{\sin(3000t)}_{1} + 10\underbrace{\sin(2000\pi t)}_{\approx 1}\Big| \approx 15\ \text{rad}$$

e) $BG = 2B(\beta + 1) = 2\cdot 1k \times (12{,}388 + 1) \approx 26{,}8\ \text{kHz}$

Haftaya 11'de quiz

Periyodik mesaj işaretinin açı modülasyonu

$$y(t) = A_c \cos\left(2\pi f_c t + \phi(t)\right)$$

$$\hookrightarrow \begin{cases} = k_p\, m(t) & \text{PM} \\ = 2\pi k_f \int m(\tau)\, d\tau & \text{FM} \end{cases}$$

$m(t)$ $T_m = \frac{1}{f_m}$ periyodik $\Rightarrow$ $\phi(t)$ $T_m = \frac{1}{f_m}$ ile periyodik

$$\hat{y}(t) = A_c\, e^{j2\pi f_c t} \underbrace{e^{j\phi(t)}}_{x(t)} \rightarrow y(t) = \mathrm{Re}\{\hat{y}(t)\}$$

$$x(t) = e^{j\phi(t)} = \sum_n \underbrace{c_n}_{|c_n| e^{j\angle c_n}} e^{j2\pi n f_m t}$$

$$c_n = \frac{1}{T_m} \int_{T_m} \underbrace{e^{j\phi(t)}}_{x(t)} \cdot e^{-j2\pi n f_m t}\, dt$$

$$\hat{y}(t) = A_c \cdot e^{j2\pi f_c t} \underbrace{\sum_{n=-\infty}^{+\infty} |c_n|\, e^{j[(2\pi n f_m t) + \angle c_n]}}_{x(t) = e^{j\phi(t)}}$$

$$y(t) = \mathrm{Re}\{\hat{y}(t)\}$$

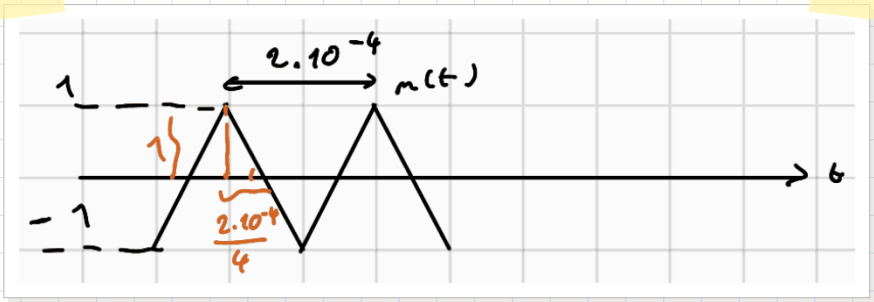
$$y(t) = A_c \sum_{n=-\infty}^{+\infty} |c_n| \cos\left(2\pi(f_c + n f_m)t + \angle c_n\right)$$

$$\downarrow \quad P_y = \frac{A_c^2}{2} \sum_{n=-\infty}^{+\infty} |c_n|^2$$

orj. s. 9

Örnek

Lathi, Ex 5.3



orijinal çizim, s. 9

eğim $\frac{\text{karşı}}{\text{komşu}} = \frac{1}{2 \cdot 10^{-4}/4} = 20000$

a) BW_{FM} =? BW_{PM} =?
if $k_f = 10^5$, $k_p = 5\pi$
b) Repeat a) if the amplitude of $m(t)$ is doubled

c) Repeat a) if $m(t)$ is time expanded by a factor of 2, i.e, the period of $m(t)$ is $4 \cdot 10^{-4}$ s

$$T_m = 2 \cdot 10^{-4}\ \text{s} \rightarrow f_m = \frac{1}{T_m} = 5\ \text{kHz}$$

$$|c_n| = \begin{cases} \dfrac{4}{\pi^2 n^2} & n \text{ tek} \\ 0 & n \text{ çift} \end{cases}$$

$\rightarrow$ $n = 3$ ise 3. harmonik %11 of the fundamental (f_m) $\hookrightarrow$ güce katkısı %1,21
$n = 5$ 5. harmonik %4 of the fundamental (f_m) $\rightarrow$ %0,16

$c_3 = \dfrac{4}{9\pi^2} = \dfrac{c_1}{9} \approx 0{,}11\, c_1$
yani: 3. harmonik birincinin %11'i;
$|c_3|^2 = 0{,}0121\,|c_1|^2$ (= 1. harmoniğin %1,21'i)

Efektif BG'ni $n = 3$. harmonik belirler

$$B = 3f_m = 15\ \text{kHz}$$

a) FM: $\Delta f = k_f \cdot m_p = 100\,000 \times 1 = 100$ kHz

$$\rightarrow \beta = \frac{\Delta f}{B} = \frac{100k}{15k} = 6{,}66$$

$$\Rightarrow BG_{FM} = 2B(\beta + 1) = 2 \cdot 15k\left(\frac{100}{15} + 1\right) = 230\ \text{kHz}$$

PM: $\Delta f = \dfrac{k_p}{2\pi} m_p'$, $\quad m_p' = \max|\dot{m}(t)| = 20\,000$

$$\Rightarrow \Delta f = \frac{5\pi}{2\pi} \times 20\,000 = 50\ \text{kHz} \rightarrow \beta = \frac{\Delta f}{B} = \frac{50}{15}$$

orj. s. 10

$$BG_{PM} = 2B(\beta + 1) = 2 \times 15k\left(\frac{50}{15} + 1\right)$$
$$= 130\ \text{kHz}$$

b) FM $m_p = 2 \rightarrow \Delta f = 200$ kHz $\Rightarrow BG_{FM} = 2B(\beta + 1)$
$= 2\left(\underset{200}{\Delta f} + \underset{15}{B}\right)$
$= 430$ kHz

PM $m_p' = 40\,000 \rightarrow \Delta f = 100$ kHz $\Rightarrow BG_{PM} = 2(\Delta f + B)$
$= 230$ kHz

Bant genişliği iki katına çıktı

c) $T_m = 4 \cdot 10^{-4}$ s $\rightarrow f_m = \dfrac{1}{T_m} = 2{,}5$ kHz

$$BG_{eff} = B = 3f_m = 7{,}5\ \text{kHz}$$

FM: $\Delta f = k_f \cdot m_p = 100$ kHz

$$\beta = \frac{\Delta f}{B} = \frac{100}{7{,}5}$$

$$BG_{FM} = 2\left(\underset{100k}{\Delta f} + \underset{7{,}5}{B}\right) = 215\ \text{kHz}$$

$$\text{PM: } \Delta f = \frac{k_p}{2\pi} \cdot m_p'$$

$$m_p' = \max|\dot{m}(t)| = 10\,000$$

$$\Delta f = \frac{5\pi}{2\pi} \cdot 10\,000 = 25 \text{ kHz}$$

$$\Rightarrow BG_{PM} = 2\underbrace{(\Delta f}_{25} + \underbrace{B}_{7,5}) = 65 \text{ kHz}$$

PM bant genişliği yarıya indi, frekans yarıya inince

FM zamandaki genişliğe pek duyarlı değil