Analog Haberleşme

İşaretler, güç/enerji, temel işaretler ve sistemler

Orijinal not: Ders2.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1

Ders 2

Enerji:

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$E_x < \infty \Rightarrow$ Enerji işaretidir

Güç:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

$0 < P_x < \infty \Rightarrow$ Güç işaretidir

$$x(t) = \begin{cases} A\cos(2\pi f_0 t + \theta) \\ A\sin(2\pi f_0 t + \theta) \end{cases} \longrightarrow \quad P_x = \frac{A^2}{2}$$

$T_0 = \frac{1}{f_0}$ periyottur.

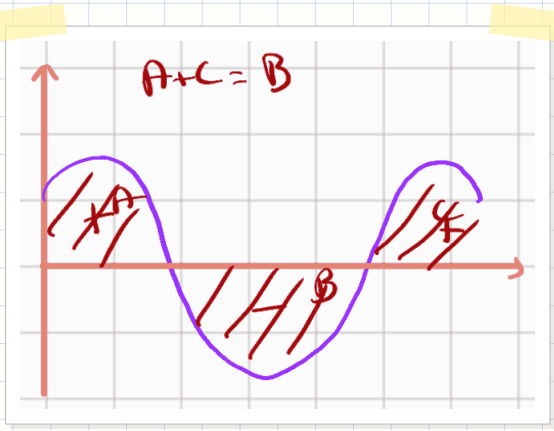
Periyodik $\Rightarrow P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2\, dt$

ispat

$$P_x = \frac{1}{T_0} \int_{T_0} A^2 \cos^2(2\pi f_0 t + \theta)\, dt = \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} \left[1 + \cos(4\pi f_0 t + 2\theta)\right] dt$$

$$= \frac{A^2}{2T_0} \left[T_0 + \underbrace{\int_{-T_0/2}^{T_0/2} \cos(4\pi f_0 t + 2\theta)\, dt}_{0} \right]$$

$*$ sinüsoidal işaretlerin bir periyot boyunca integrali 0'dır.



orijinal çizim, s. 1

$$P_x = \frac{A^2}{2}$$

Örnek

Lathi'den örnek

$x(t) = c_1 \cos(2\pi f_1 t + \theta_1) + c_2 \cos(2\pi f_2 t + \theta_2), \quad f_1 \neq f_2$

$P_x = ?$

Bir işaretin periyodunu bileşenlerinin periyodlarının en küçük ortak katlarıyla bulunabilir

orj. s. 2

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$= \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} c_1^2 \cos^2(2\pi f_1 t + \theta_1)\, dt}_{c_1^2/2}$$

$$+ \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} c_2^2 \cos^2(2\pi f_2 t + \theta_2)\, dt}_{c_2^2/2}$$

$$+ \lim_{T\to\infty} \frac{1}{T} 2c_1c_2 \int_{-T/2}^{T/2} \cos(2\pi f_1 t + \theta_1) \cos(2\pi f_2 t + \theta_2)\, dt$$

$$\lim_{T\to\infty} \frac{1}{T} c_1c_2 \int_{-T/2}^{T/2} \Big[\underbrace{\cos\big(2\pi(f_1 + f_2)t + \theta_1 + \theta_2\big)}_{0}$$

$$+ \underbrace{\cos\big(2\pi(f_1 - f_2)t + \theta_1 - \theta_2\big)}_{0} \Big] dt$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\Big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big]$$

çünkü sinüsoidal ↘ burdan bu terim gelir

$$\Rightarrow \quad P_x = \begin{cases} \dfrac{c_1^2}{2} + \dfrac{c_2^2}{2}, & f_1 \neq f_2 \\ \dfrac{c_1^2}{2} + \dfrac{c_2^2}{2} + c_1c_2\cos(\theta_1 - \theta_2), & f_1 = f_2 \end{cases}$$

Sonuçlar

$x(t) = x_1(t) + x_2(t) \Rightarrow P_x \neq P_{x_1} + P_{x_2}$

compact fourier açılımı ↙

$$x(t) = \sum_{n=1}^{\infty} c_n \cos(2\pi f_n t + \theta_n) \Rightarrow P_x = \frac{1}{2}\sum_{n=1}^{\infty} c_n^2 \qquad i \neq j,\ f_i \neq f_j$$

(Parseval teoremi)

Örnek

$x(t) = Ae^{j2\pi f_0 t}$

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{\left|Ae^{j2\pi f_0 t}\right|^2}_{A} dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |A|^2\,dt = |A|^2$$

$z = x + jy = |z|e^{j\theta}$

$|z| = \sqrt{x^2 + y^2}$

$\theta = \arctan\frac{y}{x}$

$|z_1 \cdot z_2| = |z_1||z_2|$

$$\left|Ae^{j2\pi f_0 t}\right| = |A|\ \underbrace{\left|e^{j\overbrace{2\pi f_0 t}^{\alpha}}\right|}_{\cos(\alpha)+j\sin(\alpha)}$$

orj. s. 3

Bazı önemli işaretler

Sinüsoidal işaretler

$$x(t) = \underbrace{A}_{\substack{\text{Genlik}\\(\text{Amplitude})}} \cos\left(2\pi \overbrace{f_0}^{\substack{\text{frekans (Hz}=\frac{1}{s})}} t + \underbrace{\theta}_{\substack{\text{Faz (rad)}\\(\text{phase})}}\right)$$

A güç veya enerji işaretin genliği ile orantılıdır

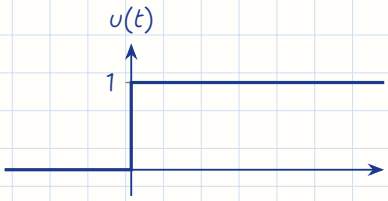
Kompleks üstel işaretler

$$x(t) = A\,e^{j(2\pi f_0 t + \theta)}$$

Birim Basamak İşareti

(Unit step signal)

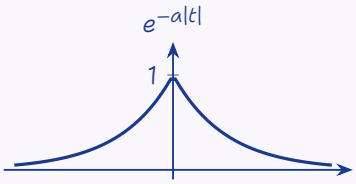
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

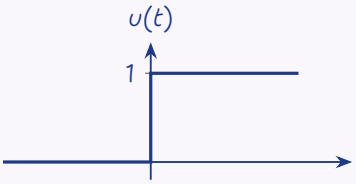


$a > 0 \qquad u(at) = u(t)$

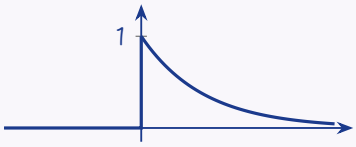
Örnek

$x(t) = e^{-a|t|}\,u(t), \quad a > 0$





$x(t) = e^{-a|t|}u(t)$



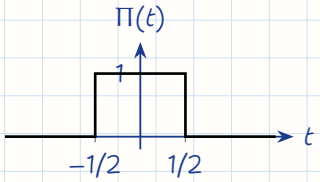
$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

orj. s. 4

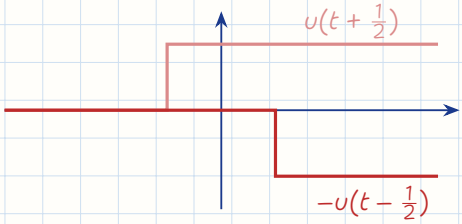
Dikdörtgen Darbe işareti

(Rectangular pulse Signal)

$$\text{rect}(t) = \Pi(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{Diğer} \end{cases}$$

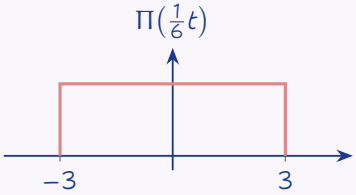


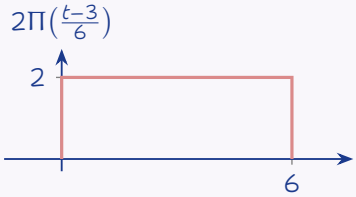
$$\Pi(t) = u\left(t + \tfrac{1}{2}\right) - u\left(t - \tfrac{1}{2}\right)$$

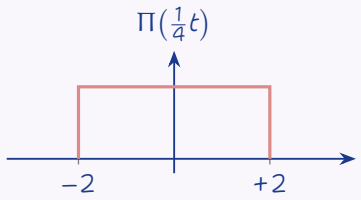


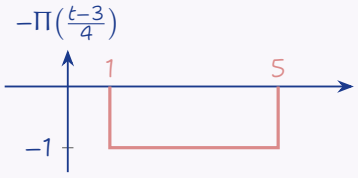
Örnek

$x(t) = 2\,\Pi\left(\underbrace{\tfrac{t-3}{6}}_{\frac{1}{6}(t-3)}\right) - \Pi\left(\underbrace{\tfrac{t-3}{4}}_{\frac{1}{4}(t-3)}\right)$ çiziniz

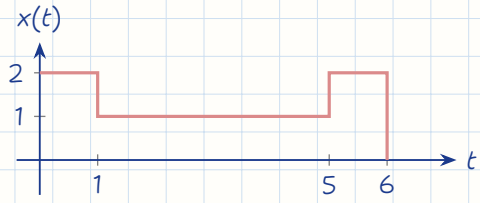




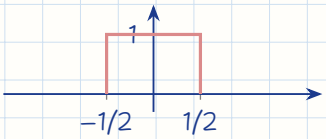




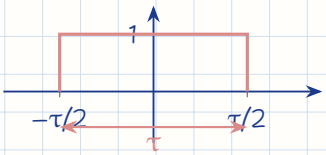
orj. s. 5



* $\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{diğer} \end{cases}$

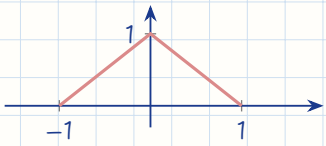


$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & \text{diğer} \end{cases}$



Üçgen darbe işareti

(Triangular Pulse Signal)



$$\Lambda(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ -t+1 & 0 \le t \le 1 \\ 0 & \text{Diğer} \end{cases}$$

Doğrunun denklemi

$$y = \overbrace{m}^{\text{eğim}}\, t + n$$

$t_1, y_1 \rightarrow \qquad t_2, y_2 \rightarrow$

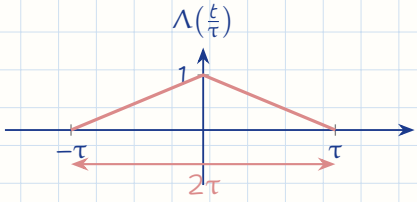
$$\Lambda(t) = \begin{cases} 1 - |t|, & \underbrace{|t| \le 1}_{-1 \le t \le 1} \\ 0, & \text{diğer} \end{cases}$$

$$\Lambda(t) = \Pi(t) \circledast \Pi(t) \quad \longrightarrow \quad \boxed{\tau\,\Lambda\left(\frac{t}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) \circledast \Pi\left(\frac{t}{\tau}\right)}$$

$$x(t) \circledast y(t) = \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} x(t-\tau)\, y(\tau)\, d\tau$$

orj. s. 6

* $\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau}, & \frac{|t|}{\tau} \le 1 \\ 0, & \text{diğer} \end{cases} \qquad : |t| \le \tau \;:\; -\tau \le t \le \tau$

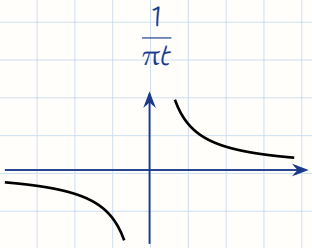


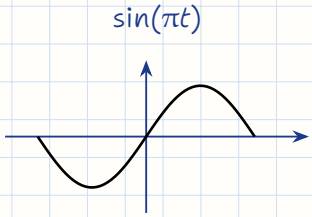
konvülasyon sonucunda elde edilen işaretin genişliği konvülasyona giren işaretlerinin genişliğinin toplamıdır.

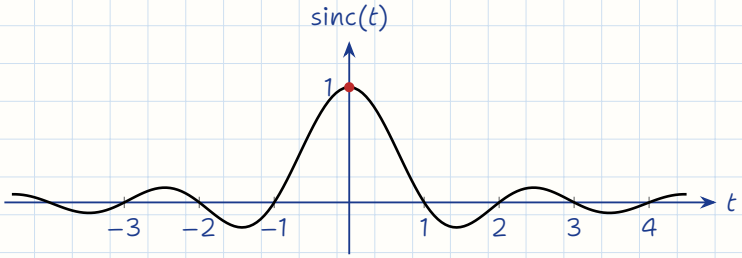
Sinc Darbe işareti

(Sinc Pulse Signal)

$$\mathrm{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$







$$* \quad (t = k) \in \mathbb{Z} - \{0\} \Rightarrow \mathrm{sinc}(k) = \frac{\sin(k\pi)}{k\pi} = 0$$

$$t = 0 \Rightarrow \mathrm{sinc}(0) = \frac{\sin(0)}{0} = \frac{0}{0}$$

$$\text{Hopital: } \lim_{t \to 0} \frac{\pi \cos(\pi t)}{\pi} = 1$$

$$\mathrm{sinc}(-t) = \mathrm{sinc}(t)$$

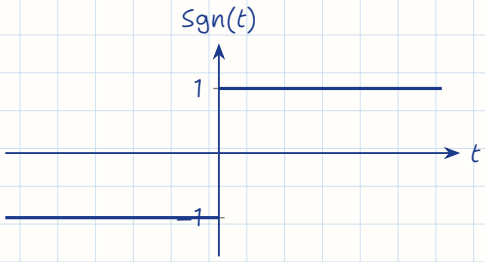
orj. s. 7

İşaret fonksiyonu

(Sign / Signum function)[1]

$$\mathrm{Sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases}$$

[1] Claude notu: Aşağıdaki Sgn(t) tanımında üçüncü satır (0, t = 0) orijinalde kırmızı kalemle çizilerek iptal edilmiş.

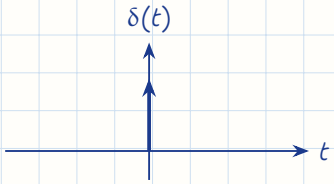


$$\mathrm{Sgn}(t) = u(t) - u(-t)$$

* $u(t) = \frac{1}{2}(1 + \mathrm{Sgn}(t))$

Dürtü işareti

(Impulse (Delta) Signal)



$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

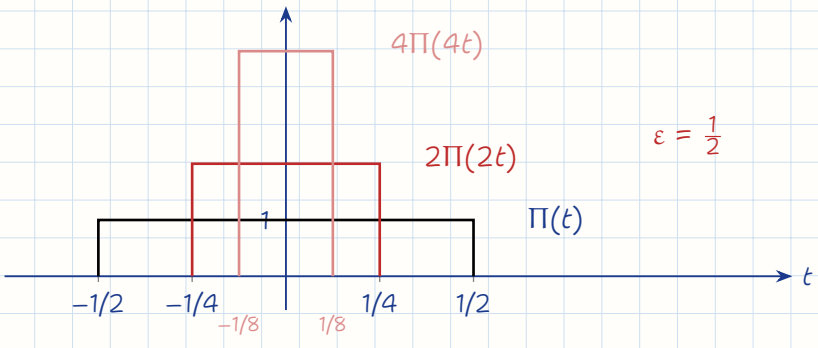
$$\text{Alan} = \text{Şiddet} = \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$[2]

$$\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt = f(0)$$
(sifting property)

← $f(t)$ $t = 0$'da sürekliyse

orj. s. 8 sinc de bu koşulu sağlıyor

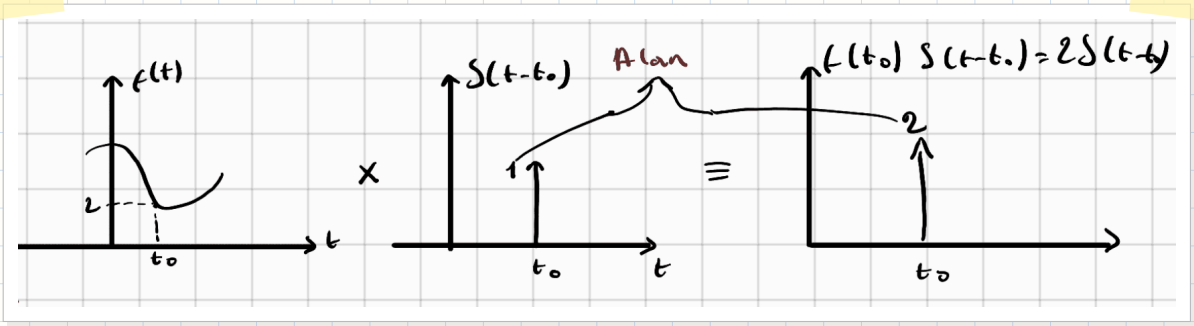
* $\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Pi\left(\frac{t}{\varepsilon}\right)$



Özellikleri: sifting/sampling property

* 1) $f(t)\,\delta(t - t_0) = f(t_0)\,\delta(t - t_0), \qquad t_0 = \text{sabit}$

[2] Claude notu: Orijinalde kutudaki integralden bu satırın sonundaki "= 1" ifadesine pembe bir ok çizilmiş; okun üzerindeki küçük not güçlükle okundu (muhtemelen "1 ise").

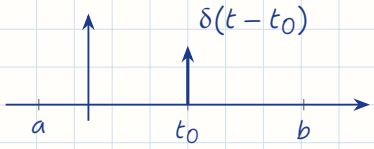


orijinal çizim, s. 8

* 2) $f(t)$ $t = t_0$'da sürekli olmak üzere

$$\int_a^b f(t)\,\delta(t-t_0)\,dt = \begin{cases} f(t_0) & a < t_0 < b \\ 0 & \text{diğer} \end{cases}$$

$$f(t) = 1 \longrightarrow \int_a^b \delta(t-t_0)\,dt = \begin{cases} 1 & 0 < t_0 < b^3 \\ 0 & \text{diğer} \end{cases}$$



$$f(t_0)\,\delta(t-t_0)$$

$$f(t_0)\underbrace{\int_a^b \delta(t-t_0)\,dt}_{1,\ a<t_0<b}$$

orj. s. 9

3) $\forall a \neq 0 \qquad \delta(at) = \dfrac{1}{|a|}\,\delta(t)$

$$a = -1 \Rightarrow \delta(-t) = \frac{1}{|-1|}\,\delta(t)$$
$$\Rightarrow \delta(-t) = \delta(t) \quad \text{çift fonksiyondur.}$$

4) $f(t) \circledast \delta(t) = f(t)$

* $f(t) \circledast \delta(t-t_0) = f(t-t_0)$

$$x(t) \circledast y(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau$$

$$f(t) \circledast \delta(t) = \int_{-\infty}^{+\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$
$$= \int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau-t)\,d\tau$$
$$= f(t)$$

[3] Claude notu: muhtemelen $a < t_0 < b$ olmalı.

$$f(t) \circledast \delta(t-t_0) = \int_{-\infty}^{+\infty} f(\tau)\, \underbrace{\delta(t-\tau-t_0)}_{\delta\left(\tau - \underbrace{(t-t_0)}_{\text{sabit}}\right)}\, d\tau$$

$$= f(t-t_0)$$

5) $$\int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \equiv u(t)$$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \xrightarrow{\frac{d}{dt}} \frac{d}{dt} u(t) = \delta(t)$$

orj. s. 10

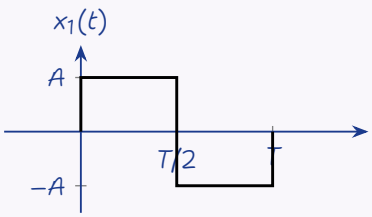
Ortogonal (Dik) / Ortonormal işaretleri

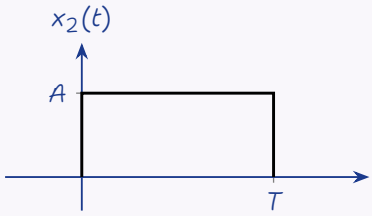
Eğer $\psi_1(t)$, $\psi_2(t)$, ... $\psi_n(t)$ işaretleri $[t_1, t_2]$ aralığında

$$\int_{t_1}^{t_2} \psi_m(t)\, \psi_n^*(t)\, dt = \begin{cases} 0 & m \neq n \\ E_m & m = n \end{cases}$$

sağlıyorsa ortagonaldır ilaveten $\forall m\ E_m = 1$ ise işaretler ortonormaldır

Örnek





$$\int_0^T x_1(t) \cdot x_2(t)\, dt = \underbrace{\int_0^{T/2} A^2\, dt}_{A^2 \frac{T}{2}} + \underbrace{\int_{T/2}^{T} A(-A)\, dt}_{-A^2\left(T - \frac{T}{2}\right)}$$

$$= 0 \quad \Rightarrow \text{ Ortagonel}$$

$$E_1 = \int_0^T x_1^2(t)\, dt$$

$$= \int_0^T A^2\, dt = A^2 T = E_2$$

orj. s. 11

Ortonormal olması için iki işaretin genlikleri $\sqrt{A^2 T}$ ye bölmemiz lazım. Böylece enerjileri 1 olur.

$$x(t) \rightarrow E_x$$

$$x_N(t) = \frac{x(t)}{\sqrt{E_x}} \rightarrow E_{x_N} = \int_{-\infty}^{\infty} x_N^2(t)\, dt$$

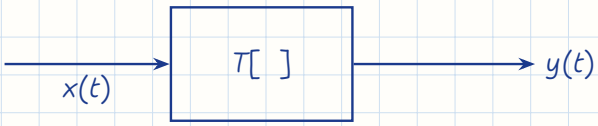
$$= \frac{1}{E_x} \underbrace{\int_{-\infty}^{\infty} x(t)^2\, dt}_{E_x}$$

$$= 1$$

Sistemler

Sistem, bir düzenektir.

$$y(t) = T[x(t)]$$



* Sürekli zamanlı s.

$$y(t) = \frac{d}{dt}x(t)$$

× Ayrık zamanlı s.

$$y[n] = x[n] - x[n-1]$$

*- Lineer

× Lineer olmayan

α, β skaler

$$y_1(t) = T[x_1(t)]$$
$$y_2(t) = T[x_2(t)]$$

$$x_3(t) = \alpha x_1(t) + \beta x_2(t)$$
$$\rightarrow\ y_3(t) = T[x_3(t)]$$
$$= T[\alpha x_1(t) + \beta x_2(t)]$$
$$= \alpha T[x_1(t)] + \beta T[x_2(t)]$$
$$= \alpha y_1(t) + \beta y_2(t)$$

orj. s. 12

Eğer bir sistemin $x_1(t)$ ve $x_2(t)$ ye cevabını biliyorsanız sonsuz tane $x_3(t)$ x_1 ve x_2'nin linear kombinasyonu cinsinden yazılabilir.

$y(t) = \frac{d}{dt}x(t)$ lineer

$y(t) = x^2(t)$ lineer değil

* Zamanla Değişen S.
(Time varient)

× Zamanla Değişmeyen S.
(Time Inveriant)

$\forall t,\ x(t) \qquad y(t) = T[x(t)]$

$$T[x(t - t_0)] = y(t - t_0)$$

$\Rightarrow$ Zamanla Değişmen (TI)

Örnek

$y(t) = t\,x(t)$

$$x_1(t) \rightarrow y_1(t) = t\,x_1(t)$$
$$x_2(t) = x(t - t_0) \rightarrow y_2(t) = t\,x_2(t) = t\,x_1(t - t_0)$$
$$y_1(t - t_0) \overset{\neq}{=} (t - t_0)\,x_1(t - t_0)$$

$\Rightarrow$ Zamanla Değişen bir işaret

Ödev $y(t) = x(t+1)$ Zamanla değişmeyen sistem

$$x_1(t) \to y_1(t) = x_1(t+1)$$
$$x_2(t) = x_1(t+1-t_0) \to y_2(t) = x_2(t) = x_1(t+1-t_0)^4$$
$$y_1(t-t_0) \overset{''}{=} x_1(t+1-t_0)$$

Zamanla değişmeyen sistem

* Lineer + Zamanla değişmen S = LZD sistem
(Lineer Time Inavariant (LTI))

orj. s. 13

Nedensel (Casual) Sistem

$$y(t_0) = T[x(t),\ t \le t_0] \Rightarrow \text{Nedensel sistem}$$

Çıkışı sadece şimdiki ve geçmişteki ana bağlı çıkış olan sistemdir. Çıkış geleceğe bağlı değildir.

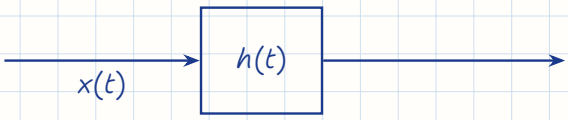
Örnek



$D \ge 0$ için nedensel

Dürtü Cevabı (Impulse Response)

$$\text{LZD sistem} \Rightarrow y(t) = T[\delta(t)] = \underbrace{h(t)}_{\substack{\text{sistemin}\\ \text{dürtü}\\ \text{cevabı}}}$$



$$y(t) = x(t) \circledast h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

LZD sistemin nedensel olması için gerek ve yeter şart $h(t) = 0,\ t < 0$

ispatı nedir??

[4] Claude notu: muhtemelen $x_2(t) = x_1(t-t_0)$ ve $y_2(t) = x_2(t+1) = x_1(t+1-t_0)$ olmalı.