Analog Haberleşme

Fourier serisi açılımları ve spektrum

Orijinal not: Ders3.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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orj. s. 1

Genelleştirilmiş Fourier Serisi Açılımı

Hatırlatma

The signals $\psi_1(t),\ \psi_2(t),\ \ldots,\ \psi_N(t)$ ortagonal over $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} \psi_m(t)\,\psi_n^*(t)\,dt = \begin{cases} 0, & m \neq n \\ E_m, & m = n \end{cases}$$

In addition, if $E_m = 1\ \forall m$ the signals are orthonormal.

Herhangi bir $x(t)$ işaretinin bu N ortogonal (dik) işaret ile yaklaşımını elde edelim

$$\begin{aligned} x(t) &\cong c_1\psi_1(t) + c_2\psi_2(t) + \cdots + c_N\psi_N(t) \\ &= \sum_{n=1} c_n\psi_n(t), \quad t_1 \le t \le t_2 \end{aligned}$$

$$\text{Hata işareti (error signal)} = e(t) = x(t) - \sum_{n=1}^{N} c_n\psi_n(t)$$

$$\text{Hatanın enerjisi (energy of error)} = E_e = \int_{t_1}^{t_2} |e(t)|^2\,dt$$

Eğer yaklaşımın katsayıları

$$c_n = \frac{1}{E_n}\overbrace{\int_{t_1}^{t_2} x(t)\,\psi_n^*(t)\,dt}^{\text{korelasyon}}; \quad n = 1, \ldots, N$$

olarak seçilirse hatanın enerjisi minimum olur.

Ortagonal işaret kümesi tam (complete) ise hatanın enerjisi sıfıra gider ve yaklaşıklık eşitlik olur.

$$x(t) = c_1\psi_1(t) + c_2\psi_2(t) + \cdots + c_N\psi_N(t) + \cdots = \sum_{n=1}^{\infty} c_n\psi_n(t), \quad t_1 \le t \le t_2$$

↪ Genelleştirilmiş F.S açılımı (Generalized FS expansion)

orj. s. 2

Şekil: Bu sayfa tahtaya yazılmış notların fotoğrafıdır (eğik çekilmiş); içerik aşağıda aktarılmıştır.
(orijinal notta, s. 2)

Eğer yaklaşımın katsayıları

$$C_n = \frac{1}{E_n}\int_{t_1}^{t_2} x(t)\,\psi_n^*(t)\,dt\,; \quad n = 1,\ldots,N$$

olarak seçilirse hatanın enerjisi minimum olur.

Ortogonal işaret kümesi tam (complete) ise hatanın enerjisi sıfıra gider ve yaklaşıklık eşitlik olur:

$$x(t) = C_1\psi_1(t) + C_2\psi_2(t) + \cdots + C_N\psi_N(t) + \cdots = \sum_{n=1}^{\infty} C_n\psi_n(t), \quad t_1 \le t \le t_2$$

↪ Genelleştirilmiş FS açılımı (Generalized FS expansion)

($x(t)$ işaret ailesi için $N \to \infty$ $E_e \to 0$ ise $\overbrace{\{\psi_n(t)\}}^{\text{Basis functions (Temel fonksiyonlar)}}$ ailesi tamdır. Bu durumda $x(t)$ işaret ailesi içindeki bütün işaretleri bu temel fonksiyonlar ile ifade etmek mümkündür.)

Parseval teoremi:

$$E_x = C_1^2E_1 + C_2^2E_2 + \cdots + C_N^2E_N + \cdots = \sum_{n=1}^{\infty} C_n^2E_n$$

orj. s. 3

Trigonometrik FS açılımı

$\forall t \;\; x(t + T_0) = x(t)$

F. Serisinin mevcut olma şartları:

1) $x(t)$ bir periyoda boyunca mutlak integrali alınabilir olmalıdır:

$$\int_{T_0} |x(t)|\,dt < \infty$$

(fourier serisi vardır, her noktada yakınsamaya bilir) } Zayıf (weak) Dirichlet şartı

2) $x(t)$'nin bir periyodunda sonlu sayıda maksimumu, minimumu ve süreksizliği vardır. } Kuvvetli (strong) Dirichlet şartları

İkisi sağlanıyorsa fourier serisi açılımı vardır.

* Laboratuvarda üretilen işaretlerin fourier açılımı vardır

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right] \qquad t_1 \le t \le t_1 + T_0$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt, \qquad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt, \quad n = 1,2,\ldots$$
$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt, \quad n = 1,2,\ldots$$

$x(t)$ tek $\Rightarrow a_0 = a_n = 0$

$x(t)$ çift $\Rightarrow b_n = 0$

$$\forall n,m \quad \int_{T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0 \quad \nearrow \text{ ortagonaltir}$$

$$\int_{T_0} \cos(n\omega_0 t)\cdot\cos(m\omega_0 t)\,dt = \begin{cases} 0 & n \neq m \\ T_0/2 & n = m \end{cases} \quad \leftarrow \textit{ortogonal}$$

$$\int_{T_0} \sin(n\omega_0 t)\,\sin(m\omega_0 t)\,dt = \begin{cases} 0, & n \neq m \\ T_0/2 & n = m \end{cases}$$

Parseval teoremi : $P_x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

orj. s. 4

Compact FS Expansion

$$a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) = c_n\cos(n\omega_0 t + \theta_n)$$

$$c_n = \sqrt{a_n^2 + b_n^2}, \qquad c_0 = a_0$$
$$\theta_n = -\arctan\left(\frac{b_n}{a_n}\right)$$
$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n\cdot\cos(n\omega_0 t + \theta_n) \qquad t_1 \le t \le t_1 + T_0$$

Parseval Teoremi : $P_x = c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} c_n^2$

↑

Parseval teoremi : $P_x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

Fourier spektrumu

$c_n \times f = nf_0$	Genlik (Amplitude) spektrumu	Çizgi (Line) spektrum
$\theta_n \times f = nf_0$	Faz (Phase) "	

Üstel FS Expansion

(Complex/Exponential FS expansion)

$$c_n \cos(n\omega_0 t + \theta_n) = c_n \cdot \frac{1}{2}\left[e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right]$$

$$= \underbrace{\frac{1}{2}c_n e^{j\theta_n}}_{x_n} \cdot e^{jn\omega_0 t} + \underbrace{\frac{1}{2}c_n \cdot e^{-j\theta_n}}_{x_{-n}} e^{-jn\omega_0 t}$$

$$x(t) = \sum_{-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

$$x_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$\int_{T_0} \underbrace{e^{jm\omega_0 t} \cdot e^{-jn\omega_0 t}}_{e^{j(m-n)\omega_0 t}} dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

Parseval teoremi : $P_x = \sum_{n=-\infty}^{+\infty} |x_n|^2$

$e^{jn\omega_0 t}$ nin gücü 1 dir.

orj. s. 5

$$x_n = \frac{1}{2}c_n e^{j\theta_n} \qquad n = 1,2,\ldots\infty$$

$$x_{-n} = \frac{1}{2}c_n e^{-j\theta_n} \qquad n = 1,2,\ldots\infty$$

$$x_0 = c_0 = a_0$$

Üstel F. spektrumu

$$x_n = |x_n|\, e^{j\theta_n}$$

$$x_{-n} = |x_n|\, e^{-j\theta_n}$$

$$|x_{-n}| = |x_n| = \frac{1}{2}c_n$$

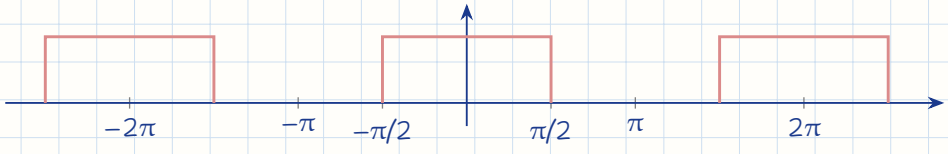
$$\angle x_n = \theta_n, \qquad \angle x_{-n} = -\theta_n$$

$|x_n| \times f = nf_0$ Genlik spektrumu

$\angle x_n \times f = nf_0$ Faz spektrumu

Örnek

Find compact FS expansion of $x(t)$
(Unipolar square wave)
Tek kutuplu kare dalgası



$\Rightarrow T_0 = 2\pi$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\,dt = \frac{1}{2\pi} \int_{-T_0/2}^{T_0/2} x(t)\,dt = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1\,dt$$
$$= \frac{1}{2\pi}\pi = \frac{1}{2} \qquad (\omega_0 = 1)$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cdot \cos(nt)\,dt \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$$
$$= \frac{2}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot \cos(nt)\,dt = \frac{1}{\pi} \frac{\sin(nt)}{n} \Big|_{-\pi/2}^{+\pi/2}$$

orj. s. 6

$$= \frac{1}{\pi n} \left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right) \right]$$

$$a_n = \frac{2}{\pi n} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ çift ise} \\ \frac{2\pi}{n}, & n = 1,5,9 \\ -\frac{2\pi}{n}, & n = 3,7,11 \end{cases}$$ [1]

$x(t)$ çift $\Rightarrow b_n = 0$

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t) + \cdots \right]$$

← ezberle

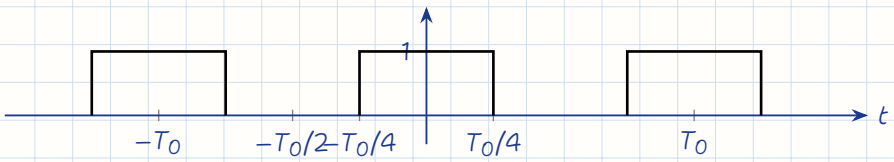
$$-\cos a = \cos(a - \pi)$$

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos(\omega_0 t) + \frac{1}{3}\cos(3\omega_0 t - \pi) + \frac{1}{5}\cos(5\omega_0 t) + \frac{1}{7}\cos(7\omega_0 t - \pi) + \cdots \right]$$

← ezberle

$$c_0 = \frac{1}{2}, \quad c_n \begin{cases} \frac{2}{\pi n} & n \text{ tek} \\ 0 & n \text{ çift} \end{cases}, \qquad \theta_n = \begin{cases} -\pi, & n = 3,7,11,\ldots \\ 0, & n \neq 3,7,11,\ldots \end{cases}$$

**



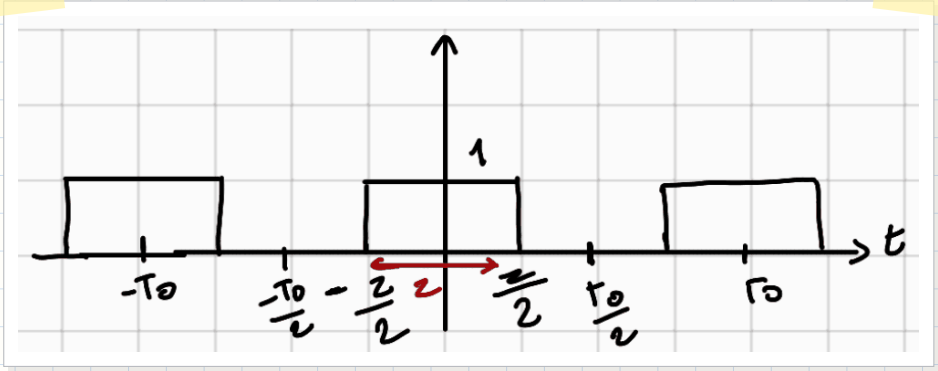
[1] Claude notu: muhtemelen $\frac{2}{\pi n}$ ve $-\frac{2}{\pi n}$ olmalı.

$$a_0 = \frac{1}{2}, \quad a_n = \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right), \quad b_n = 0$$

Burdan geçir

$$a_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} 1\cdot\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\,\frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_{-T_0/4}^{T_0/4}$$
$$= \underbrace{\frac{2}{n\omega_0 T_0}}_{2\pi}\underbrace{\left[\sin\left(n\omega_0\frac{T_0}{4}\right) - \sin\left(-n\omega_0\frac{T_0}{4}\right)\right]}_{2\sin\left(n\omega_0\frac{T_0}{4}\right)}$$
$$= \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right)$$

orj. s. 7



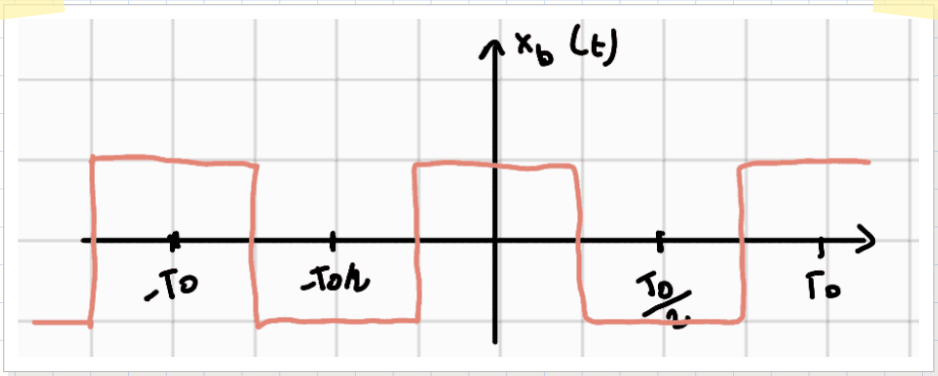
orijinal çizim, s. 7

Ödev olarak ispatlayınız

$$a_0 = \frac{\tau}{T_0}, \quad a_n = \frac{2}{\pi n}\sin\left(n\omega_0\frac{\tau}{2}\right), \quad b_n = 0$$

orj. s. 8

Bipolar square wave (Çift kutuplu kare dalga)



orijinal çizim, s. 8

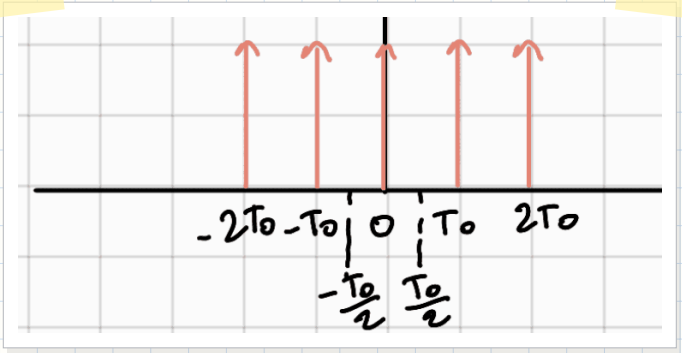
$$x_b(t) = 2x(t) - 1 = 2\left[x(t) - \frac{1}{2}\right]$$

$$x_b(t) = \frac{4}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t) + \cdots\right]$$

Örnek

Dürtü treni (Impulse train)

$$x(t) = s_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$



orijinal çizim, s. 8

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\underbrace{\int_{-T_0/2}^{T_0/2} \underbrace{x(t)}_{\delta(t)}\,dt}_{1} = \frac{1}{T_0}$$

$$\int_a^b f(t)\,\delta(t - t_0)\,dt = \begin{cases} f(t_0), & a < t_0 < b \\ 0, & \text{diğer} \end{cases}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{x(t)}_{\delta(t-0)}\,\underbrace{\cos(n\omega_0 t)}_{\cos(n\omega_0 \cdot 0)=1}\,dt = \frac{2}{T_0}$$

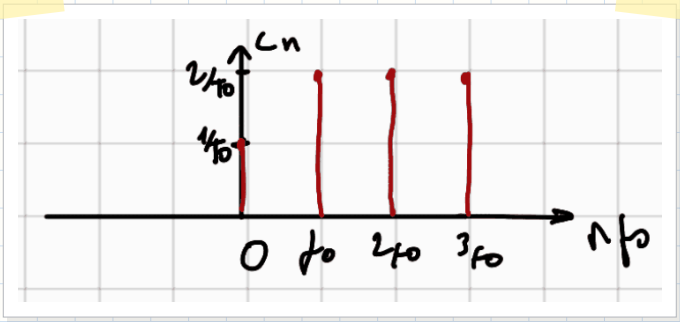
$x(t)$ çift , $b_n = 0$

$$x(t) = \underbrace{\frac{1}{T_0}}_{a_0 = c_0} + \sum_{n=1}^{\infty} \underbrace{\frac{2}{T_0}}_{a_n = c_n} \cos(n\omega_0 t)$$

$$= \frac{1}{T_0}\left[1 + 2\sum_{n=1}^{\infty} \cos(n\omega_0 t)\right]$$

$$c_0 = \frac{1}{T_0}, \quad c_n = \frac{2}{T_0}, \quad \theta_n = 0$$

orj. s. 9



orijinal çizim, s. 9

Üstel FS açılımı

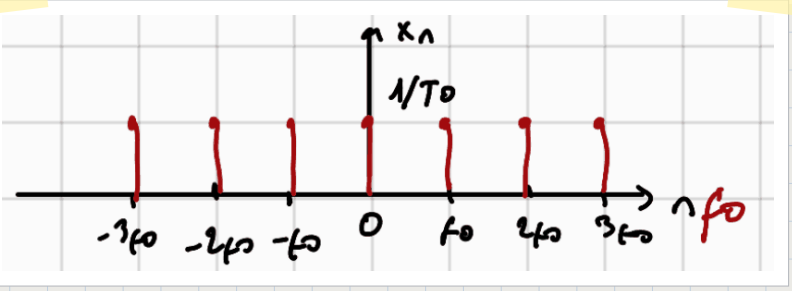
$$x_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \underset{0}{\delta(t)}\,\underbrace{e^{-jn\omega_0 t}}_{f(t)}\,dt = \frac{1}{T_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

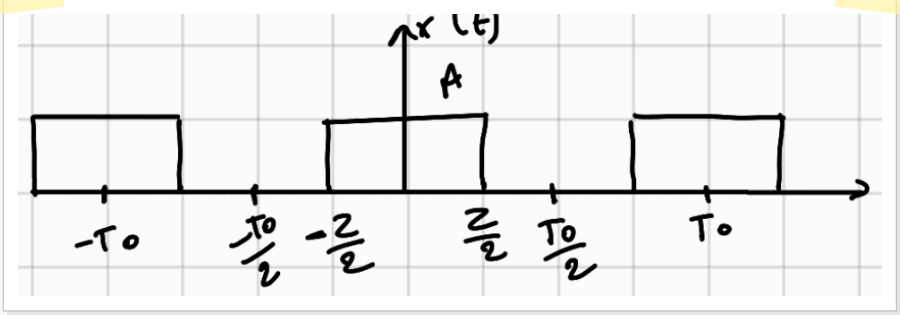
$$x(t) = s_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

$$x_n = \frac{1}{T_0}, \qquad |x_n| = \frac{1}{T_0}, \qquad \theta_n = 0$$



orijinal çizim, s. 9

Örnek



orijinal çizim, s. 9

Üstel FS açılımı

$$x_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}\,dt = \frac{A}{T_0}\,\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_{-\tau/2}^{\tau/2}$$

$$= -\frac{A}{jn\omega_0 T_0}\left[e^{-jn\omega_0\frac{\tau}{2}} - e^{+jn\omega_0\frac{\tau}{2}}\right] = \frac{A}{n\pi}\sin\left(n\omega_0\frac{\tau}{2}\right)$$

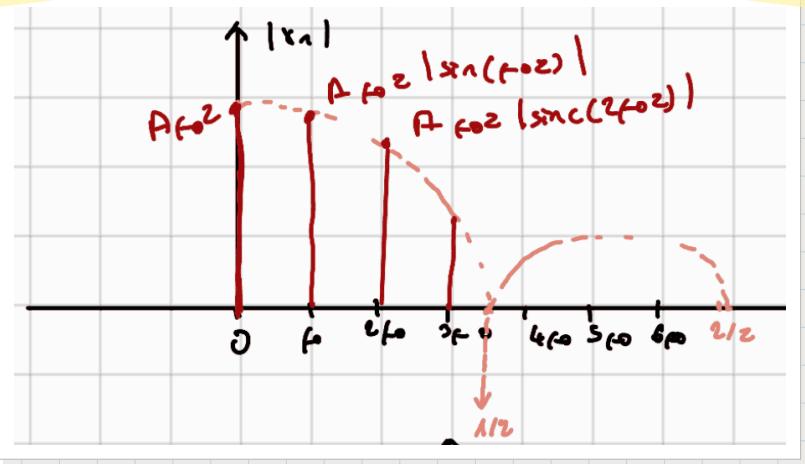
orj. s. 10

$$x_n = \frac{A}{\pi n}\sin\left(n\,2\pi f_0\frac{\tau}{2}\right) = \frac{A}{\pi n}\underbrace{\frac{\sin(n\pi f_0\tau)}{\pi n f_0\tau}}_{\mathrm{sinc}(nf_0\tau)}\cdot\pi n f_0\tau$$

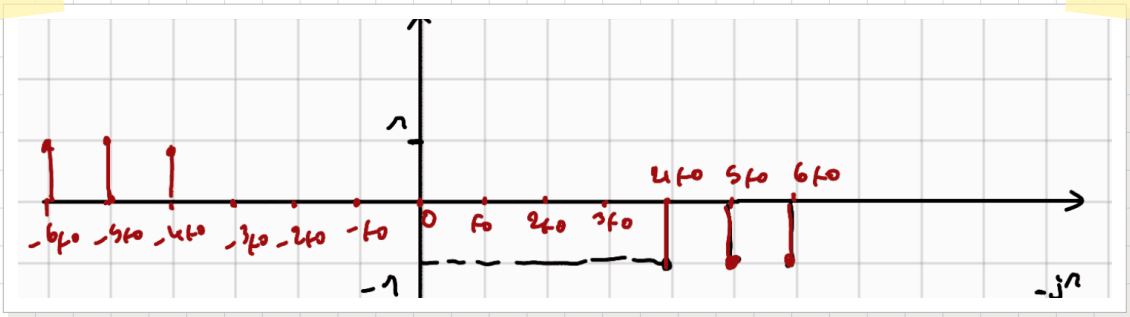
$$\boxed{x_n = Af_0\tau\,\mathrm{sinc}(nf_0\tau)}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \underbrace{Af_0\tau\,\mathrm{sinc}(nf_0\tau)}_{x_n}\cdot e^{jn\omega_0 t}$$

$\frac{T_0}{4} < \tau < \frac{T_0}{3} \rightarrow 3f_0 < \frac{1}{\tau} < 4f_0$ aralığında olsun



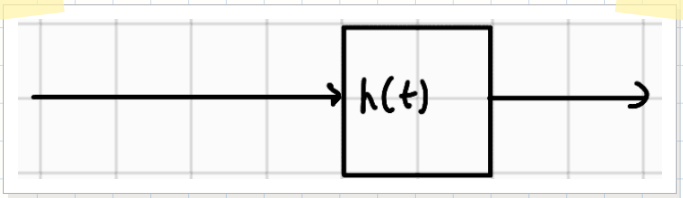
orijinal çizim, s. 10



orijinal çizim, s. 10

$$-1 = e^{-j\pi} = \underbrace{\cos\pi}_{-1} - j\underbrace{\sin\pi}_{0}$$

Ödev



orijinal çizim, s. 10

$$x(t) = \sum_n x_n e^{jn\omega_0 t} \qquad y(t) = \sum_n y_n e^{jn\omega_0 t}$$

$$y_n = x_n H(nf_0)$$
$$H(f) = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi ft}\, dt$$

∗ Example 2.2.6

orj. s. 11

> **Şekil:** Bu sayfa basılı materyal içeriyor (ör. ders slaytı); dijitalleştirilmedi.
> *(orijinal notta, s. 11)*

$$a_0 = 0, \qquad a_n = \frac{4}{\pi n}\sin\left(n\omega_0\frac{\tau}{2}\right)$$

$$T_0 = 10^{-5}, \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{10^{-5}} = 2\pi 10^5, \qquad \tau = \frac{T_0}{2}$$

$$a_n = \frac{4}{\pi n}\sin\left(n\frac{2\pi}{T_0}\cdot\frac{T_0}{4}\right)$$
$$= \frac{4}{\pi n}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = 0$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) \right] + b_n \sin(n\omega_0 t)$$

$$\sum_{n=1}^{\infty} \frac{4}{\pi n} \sin\left(\frac{n\pi}{2}\right) \cos\left(n\, 2\pi 10^5 t\right)$$