Analog Haberleşme

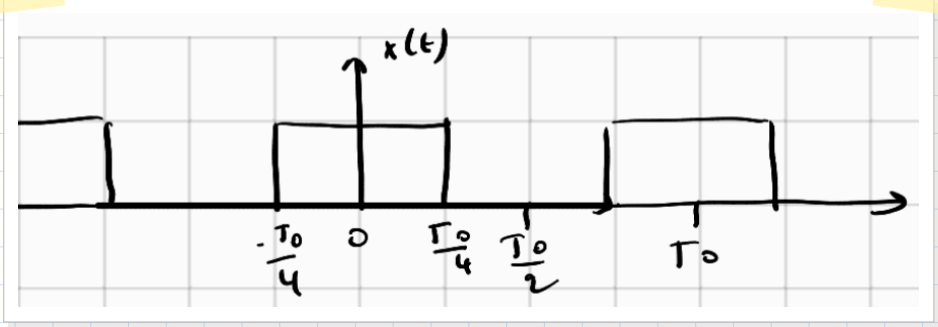
Fourier serisi, Fourier dönüşümü ve özellikleri

Orijinal not: Ders4.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1



orijinal çizim, s. 1

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t) + \cdots\right]$$

$$x_n(t) = 2x(t) - 1 = \frac{4}{\pi}\left[\downarrow\right]$$

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n e^{jn\omega_0 t}$$

$$x_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$P_x = \sum_{n=-\infty}^{+\infty} |x_n|^2$$

Örnek

$x(t) = 2\sin(100t)$ işaretinin gücünü bulunuz

$$\omega_0 = 100 = \frac{2\pi}{T_0} \rightarrow T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{100} = \frac{\pi}{50}$$

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \frac{1}{\pi/50}\int_{-\pi/100}^{\pi/100} 4\, \underbrace{\sin^2(100t)}_{\frac{1}{2}[1-\cos(200t)]}\, dt = \frac{100}{\pi}\left[t - \frac{\sin(200t)}{200}\right]_{-\pi/100}^{\pi/100}$$

$$= \frac{100}{\pi}\left\{\left[\frac{\pi}{100} - \frac{\sin\left(200 \cdot \frac{\pi}{100}\right)}{200}\right] - \left[-\frac{\pi}{100} - \frac{\sin\left(200 \cdot \frac{-\pi}{100}\right)}{200}\right]\right\}$$

$$= \frac{100}{\pi} \cdot \frac{2\pi}{100} = 2\,\text{Watt}$$

$$P_x = \frac{A^2}{2} = \frac{2^2}{2} = 2\,\text{Watt} \;//$$

$$x(t) = 2\sin(100t) = 2\,\frac{e^{j100t} - e^{-j100t}}{2j} = \underbrace{\frac{1}{j}}_{x_1} e^{j100t} \quad \underbrace{-\frac{1}{j}}_{x_{-1}} e^{-j100t}$$

$$P_x = \sum_n |x_n|^2 = \left|\frac{1}{j}\right|^2 + \left|-\frac{1}{j}\right|^2 = 1 + 1 = 2\,\text{watt}$$

$$\star \quad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \qquad \cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

orj. s. 2

Fourier Dönüşümü

- Periyodik olmayan işaretler için
- Dirichlet şartları (T_0 yerine belirli aralıkta ifadesi ?g.)

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft}\, df$$

$$x(t) \overset{F}{\longleftrightarrow} X(f) \Rightarrow X(f) = \underbrace{|X(f)|}_{\substack{\text{Genlik spektrumu} \\ \text{(magnitude)} \\ \uparrow \text{ her zaman pozitif}}} e^{j\theta_x(f)} \;\searrow \text{faz spektrumu}$$

Örnek

$x(t) = \delta(t)$

$$X(f) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi ft}\, dt$$
$$= e^{-j2\pi f \cdot 0} = 1$$

$\star\ \delta(t) \overset{F}{\longleftrightarrow} 1$ ← önemli

Örnek

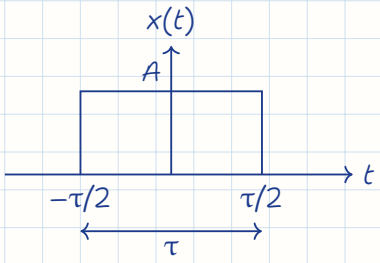
$X(f) = \delta(f)$

$$x(t) = \int_{-\infty}^{\infty} \delta(f)\, e^{j2\pi ft}\, df = e^{j2\pi \cdot 0 \cdot t} = 1$$

$$1 \overset{F}{\longleftrightarrow} \delta(f) \quad \leftarrow \textit{önemli}$$

Örnek

$x(t) = A\,\Pi\left(\frac{t}{\tau}\right)$



$$X(f) = \int_{-\tau/2}^{\tau/2} A\, e^{-j2\pi ft}\, dt$$

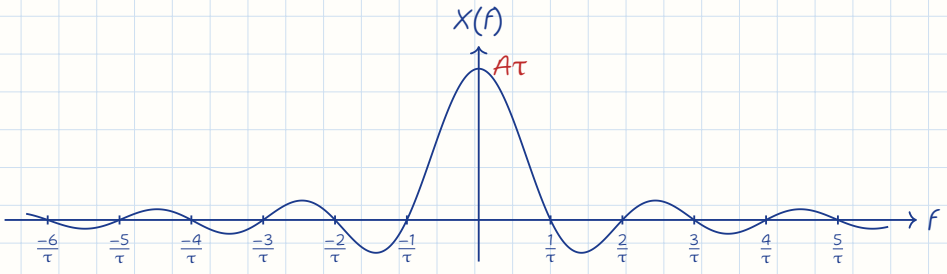
$$= A\,\frac{e^{-j2\pi ft}}{-j2\pi f}\Bigg|_{-\tau/2}^{\tau/2} = A\left[\frac{e^{-j2\pi f\tau/2} - e^{j2\pi f\tau/2}}{-j2\pi f}\right] =$$

$$X(f) = \frac{A}{\pi f \cdot \tau}\sin(\pi f\tau)\cdot\tau$$

$$X(f) = A\tau\,\mathrm{sinc}(f\tau)$$

$$x(t) = A\,\Pi\left(\frac{t}{\tau}\right) \overset{F}{\longleftrightarrow} X(f) = A\tau\,\mathrm{sinc}(f\tau) \quad \leftarrow \textit{önemli}$$

orj. s. 3

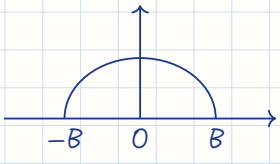


$$f\tau = k, \quad k \in \mathbb{Z} - \{0\} \Rightarrow \mathrm{sinc}(k) = 0$$

$$f = \frac{k}{\tau}$$

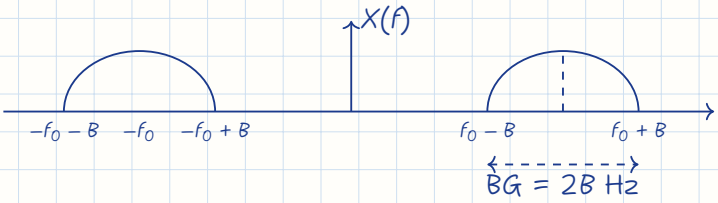
Band Genişliği (BG) (Bandwidth-BW)

Temel band (bileşenler 0 Hz'in civarında) işaretleri için genlik spektrumun sıfırdan farklı olduğu en yüksek pozitif frekansa BG denir.



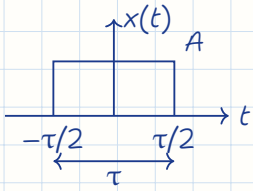
BG = B Hz

Band geçiren (Bandpass) işaretler için genlik spektrumun sıfırdan farklı olduğu pozitif frekanslar bölgesinin genişliğine BG denir.



Sinc'in teorik olarak BG= ∞

%90 BG = effective bant genişliği = BG = 1/τ



Genişlik		BG
(zaman domeninde)		(frekans domeninde)
τ	x	BG = 1/τ

Veri hızıyla bant genişliği arasında doğrudan ilişki var.

FTM

orj. s. 4

Lineerlik özelliği

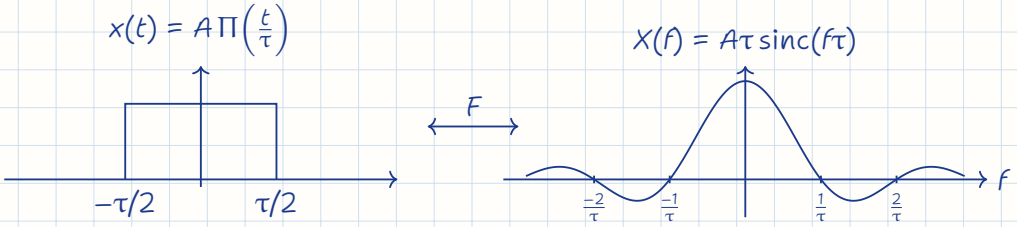
$$x_1(t) \overset{F}{\longleftrightarrow} X_1(f)$$
$$x_2(t) \overset{F}{\longleftrightarrow} X_2(f)$$

a, b sabit

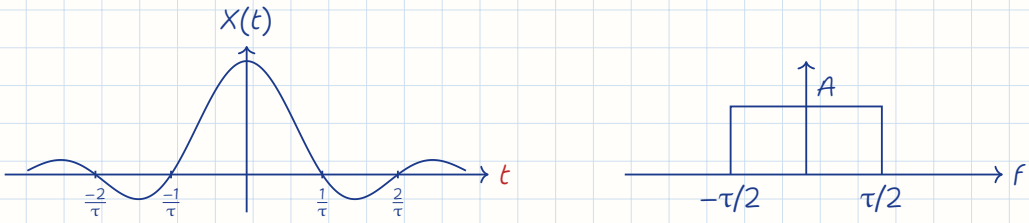
$$x_3(t) = a\,x_1(t) + b\,x_2(t) \overset{F}{\longleftrightarrow} a\,X_1(f) + b\,X_2(f) = X_3(f)$$

Dualite (Simetri) özelliği

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$
$$X(t) \overset{F}{\longleftrightarrow} x(-f)$$



$$X(t) = A\tau\,\mathrm{sinc}(t\tau) \overset{F}{\longleftrightarrow} x(-f) = A\Pi\left(\frac{-f}{\tau}\right) = A\Pi\left(\frac{f}{\tau}\right)$$



$$A\tau\,\mathrm{sinc}(t\tau) \overset{F}{\longleftrightarrow} A\Pi\left(\frac{f}{\tau}\right)$$

Zamanda öteleme özelliği (Time shifting property)

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$
$$x(t - t_0) \overset{F}{\longleftrightarrow} X(f) \cdot e^{-j2\pi f t_0}$$

İspat:

$$\mathcal{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j2\pi f t}\, dt$$

$$t - t_0 = t' \Rightarrow t = t_0 + t'$$
$$dt = dt'$$

$$= \int_{-\infty}^{+\infty} x(t') \underbrace{e^{-j2\pi f(t_0 + t')}}_{e^{-j2\pi f t_0} \cdot e^{-j2\pi f t'}} dt'$$

orj. s. 5

$$= e^{-j2\pi f t_0} \underbrace{\int_{-\infty}^{\infty} x(t')\, e^{-j2\pi f t'}\, dt'}_{X(f)}$$

Frekansta öteleme özelliği

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$
$$x(t) \cdot e^{j2\pi f_0 t} \overset{F}{\longleftrightarrow} X(f - f_0)$$

f_0 = sabit

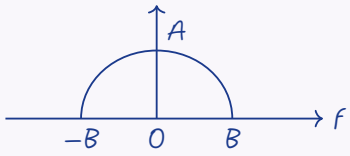
İspat:

$$\mathcal{F}\left\{x(t)\,e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} x(t)\,\underbrace{e^{j2\pi f_0 t}\cdot e^{-j2\pi f t}}_{e^{-j2\pi(f-f_0)t}}\,dt$$

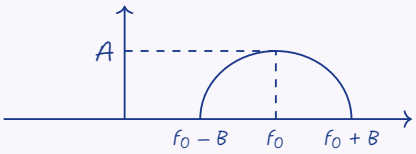
$$= \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi\underbrace{(f-f_0)}_{f'}t}\,dt$$

$$= X(f') = X(f-f_0)$$

Örnek



$$\mathcal{F}\left\{x(t)\cdot e^{j2\pi f_0 t}\right\} = X(f-f_0)$$



Örnek

$$x(t) = \cos(2\pi f_0 t) = \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

$$1 \longleftrightarrow \delta(f)$$

$$x(t)\,e^{j2\pi f_0 t} \overset{F}{\longleftrightarrow} X(f-f_0)$$

$$1\cdot e^{j2\pi f_0 t} \overset{F}{\longleftrightarrow} \delta(f-f_0) \longrightarrow$$

$$x(t) = \cos(2\pi f_0 t)$$

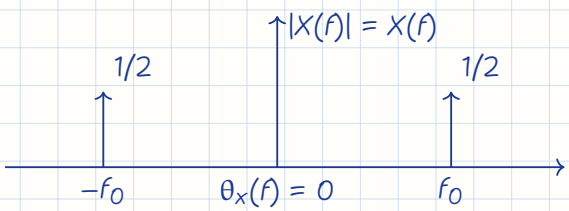
$$\updownarrow F$$

$$X(f) = \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$\cos(2\pi f_0 t) \overset{F}{\longleftrightarrow} \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] \quad \leftarrow \textbf{önemli}$$

$$X(f) = |X(f)|$$

$$\theta_x(f) = 0$$



orj. s. 6

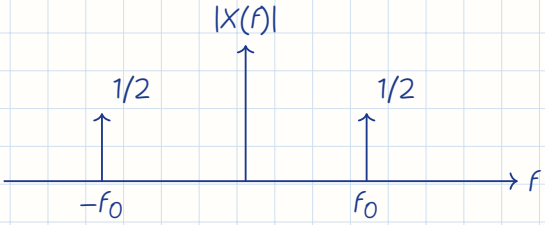
$$x(t) = \sin(2\pi f_0 t) = \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

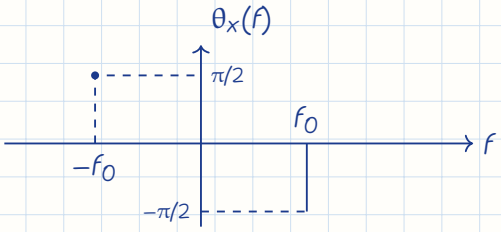
$$X(f) = \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

$$j = e^{j\pi/2}$$

$e^{\pm j\alpha} = \cos\alpha \pm j\sin\alpha$

$$X(f) = \frac{1}{2}\left[\delta(f - f_0)\, e^{-j\pi/2} + \delta(f + f_0)\, e^{j\pi/2}\right]$$





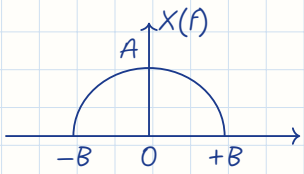
Modülasyon özelliği

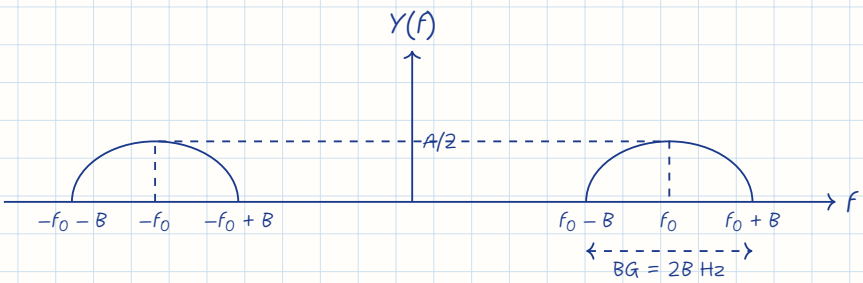
$$x(t) \overset{F}{\longleftrightarrow} X(f)$$

taşıyıcı işaret (Carrier Signal)

$$\underbrace{y(t)}_{\text{modüle edilmiş işaret}} = \underbrace{x(t)}_{\text{mesaj işareti}} \cdot \underbrace{\cos(2\pi f_0 t)}_{\frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]}$$

$$\overset{F}{\longleftrightarrow} Y(f) = \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$$





frekans bölmeli çoklama

Ölçekleme özelliği (Scaling property)

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$a = 2: \quad \underbrace{x(2t)}_{\text{zamanda daralması}} \longleftrightarrow \frac{1}{2} \underbrace{X\left(\frac{f}{2}\right)}_{\text{Frekansta genişlenir}}$$

İşaretin Genişliği x BG

orj. s. 7

Konvülasyon özelliği

$$x(t) \circledast y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$z(t) = x(t) \circledast y(t) \overset{F}{\longleftrightarrow} Z(f) = X(f) \cdot Y(f)$$

$$z(t) = x(t) \cdot y(t) \overset{F}{\longleftrightarrow} Z(f) = X(f) \circledast Y(f)$$

$$z(t) = \underbrace{x(t)}_{BG=B_1} \circledast \underbrace{y(t)}_{BG=B_2} \rightarrow \text{Genişlik}\{z(t)\} = \text{Genişlik}\{x(t)\} + \text{Genişlik}\{y(t)\}$$

$$z(t) = x(t) \cdot y(t) \Rightarrow Z(f) = \underbrace{X(f)}_{BG=B_1} \circledast \underbrace{Y(f)}_{BG=B_2}$$

$$\downarrow \quad BG = B_1 + B_2$$

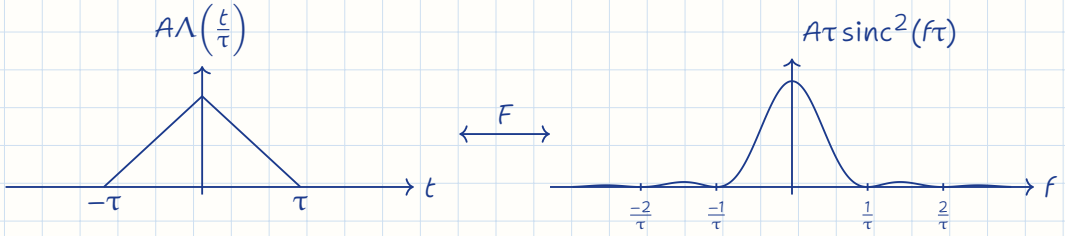
Örnek

$\Lambda(t) = \Pi(t) \circledast \Pi(t) \Rightarrow \mathcal{F}\left\{A\Lambda\left(\frac{t}{\tau}\right)\right\} = ?$

$$\mathcal{F}\{\Lambda(t)\} = \underbrace{\mathcal{F}\{\Pi(t)\}}_{\text{sinc } f} \cdot \underbrace{\mathcal{F}\{\Pi(t)\}}_{\text{sinc } f} = \text{sinc}^2(f)$$

$$\mathcal{F}\left\{A\Lambda\left(\frac{t}{\tau}\right)\right\} = A\tau\, \text{sinc}^2(f\tau)$$

$$A\Lambda\left(\frac{t}{\tau}\right) \overset{F}{\longleftrightarrow} A\tau\, \text{sinc}^2(f\tau) \quad \leftarrow \textit{önemli}$$



orj. s. 8

İntegral özelliği

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \overset{F}{\longleftrightarrow} Y(f) = \frac{X(f)}{j2\pi f} + \frac{1}{2}\overbrace{X(0)}^{\text{dc bileşeni}=X(f)|_{f=0}}\delta(f)$$

frekans arttıkça genliği azalıyor
(alçak geçiren filtre gibi davranıyor)

Örnek

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \overset{F}{\longleftrightarrow} \frac{1}{j2\pi f} + \frac{1}{2}\cdot 1\cdot\delta(f)$$

$$u(t) \overset{F}{\longleftrightarrow} \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

(bu $U(f)$? eşit değeri)
↪ bu frekans uzayında birim basamak

Örnek

$$\mathrm{Sgn}(t) = u(t) - \underbrace{u(-t)}_{a=-1}$$

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) - \frac{1}{|-1|}\left\{\frac{1}{j2\pi\frac{f}{-1}} + \frac{1}{2}\underbrace{\delta\left(\frac{f}{-1}\right)}_{=\delta(f)}\right\}$$

$$= \frac{1}{j\pi f}$$

$$\mathrm{Sgn}(t) \overset{F}{\longleftrightarrow} \frac{1}{j\pi f}$$ ← önemli

Türev özelliği

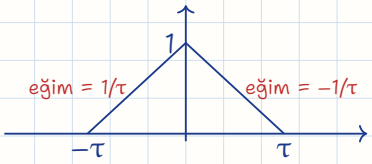
$$x(t) \overset{F}{\longleftrightarrow} X(f)$$

$$y(t) = \frac{d}{dt}x(t) \overset{F}{\longleftrightarrow} Y(f) = j2\pi f X(f) \quad \rightarrow \quad \text{frekans arttıkça işaret güçlenir (yüksek geçiren f)}$$

$$y(t) = \frac{d^n}{dt^n}x(t) \overset{F}{\longleftrightarrow} Y(f) = (j2\pi f)^n X(f)$$

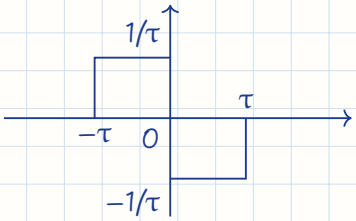
Örnek

$x(t) = \Lambda\left(\frac{t}{\tau}\right)$'nun FD'nü türev özelliğini kullanarak bulunuz



orj. s. 9

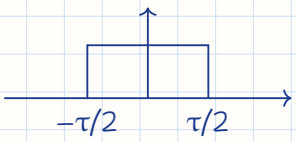
$$y(t) = \frac{d}{dt}x(t)$$



$$y(t) = \frac{d}{dt}x(t)$$

$$Y(f) = j2\pi f X(f) \rightarrow X(f) = \frac{Y(f)}{j2\pi f}$$

$$z(t) = A\,\Pi\left(\frac{t}{\tau}\right)$$



$$y(t) = z\left(t + \tfrac{\tau}{2}\right) - z\left(t - \tfrac{\tau}{2}\right)$$

$$Y(f) = Z(f)\,e^{j2\pi f\frac{\tau}{2}} - Z(f)\,e^{-j2\pi f\frac{\tau}{2}}$$

$$Y(f) = Z(f)\left[e^{j\pi f\tau} - e^{-j\pi f\tau}\right]$$

$$X(f) = \frac{Y(f)}{j2\pi f} = \frac{Z(f)}{j2\pi f}\left[e^{j\pi f\tau} - e^{-j\pi f\tau}\right]$$

$$= \frac{Z(f)}{\pi f \cdot \tau}\sin(\pi f\tau)\cdot\tau$$

$$X(f) = \underbrace{Z(f)}_{\substack{\downarrow \\ A\tau\,\mathrm{sinc}(f\tau) \\ \downarrow \\ 1/\tau}} \cdot \tau\,\mathrm{sinc}(f\tau) = \tau\,\mathrm{sinc}^2(f\tau)$$

Periyodik işaretlerin FD

$\forall t,\ x(t + T_0) = x(t)$

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n\, e^{jn\omega_0 t}$$
$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$
$$x_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

← önemli

$$e^{j2\pi f_0 t} \overset{F}{\longleftrightarrow} \delta(f - f_0)$$
$$e^{jn\omega_0 t} = e^{jn2\pi f_0 t} \overset{F}{\longleftrightarrow} \delta(f - nf_0)$$

orj. s. 10

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n\, e^{j2\pi n f_0 t} \overset{F}{\longleftrightarrow} X(f) = \sum_{n=-\infty}^{+\infty} x_n\, \delta(f - nf_0)$$

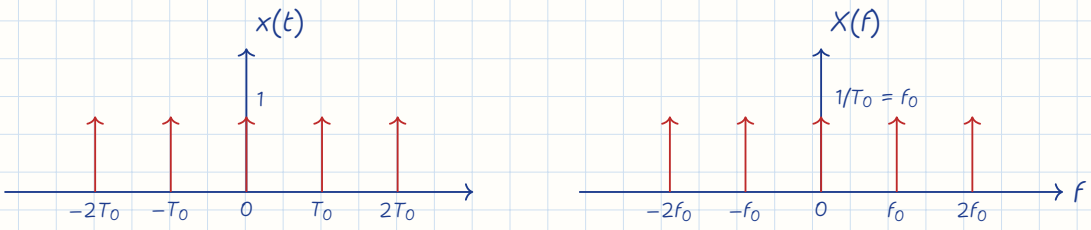
cos ve sin fourier dönüşümü içeriyor onun için mantıklı.

Örnek

$$x(t) = \delta_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} e^{j2\pi n f_0 t}$$
$$\overset{F}{\longleftrightarrow} X(f) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

Dürtü katarı

⋆ t ve f'te dürtü katarı



Parseval Teoremi

$$\star\ \underbrace{E_x}_{\text{Sınav sorusu}} = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}_{\text{Enerji yoğunluk fonksiyonu}}\, df$$

** $A\tau\,\mathrm{sinc}(t\tau)$ enerjisini bulunuz,
ama fourier dönüşümünden ? yapsanız -

$$x(t) \text{ reel} \Rightarrow X(-f) = X^*(f)$$

* $x(t)$ reel ve çift $\Rightarrow$ $X(f)$ reel ve çift $\rightarrow$ $X(-f) = X(f)$

$\swarrow$ $x^*(t) = x(t)$ $\quad\downarrow$ $x(-t) = x(t)$ $\quad\hookrightarrow$ $X^*(f) = X(f)$ ✓

* Ödev ispatla

$$x(t) \text{ sanal ve tek ise} \Rightarrow X(f) \text{ sanal ve tek}$$

$\downarrow$ $X^*(f) = -X(f)$ $\quad\hookrightarrow$ $X(-f) = -X(f)$