Analog Haberleşme

Lineer sistemlerde işaret iletimi; enerji ve güç spektral yoğunluğu

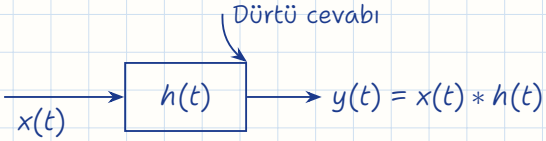
Orijinal not: Ders5.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

Lineer sistemlerden işaretlerin iletilmesi

orj. s. 1



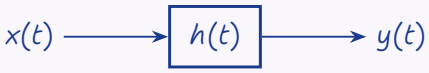
$Y(f) = X(f) \cdot H(f)$ $\quad$ ($H(f)$: Transfer fonk.)

$|Y(f)|\, e^{j\theta_y(f)} = |X(f)|\, e^{j\theta_x(f)} \cdot |H(f)|\, e^{j\theta_h(f)}$

$|Y(f)| = |X(f)| \cdot |H(f)|$

$\theta_y(f) = \theta_x(f) + \theta_h(f)$

Örnek



$x(t) = 3 + 5\cos\left(3t + \frac{\pi}{4}\right)$, $\quad H(f) = j20\pi f = 10 \cdot j2\pi f \Rightarrow h(t) = 10\,\frac{d}{dt}(\,)$

a) $x(t)$'nin periyodu? $y(t)$'nin periyodu? b) $P_x = ?$ c) $y(t) = ?$ d) $P_y = ?$

a) $x(t) = 3 + 5\cos(\underbrace{3}_{\omega_0} t + \frac{\pi}{4})$ $\quad \omega_0 = 3 = 2\pi f_0 = \frac{2\pi}{T_0} \rightarrow T_{0x} = \frac{2\pi}{3}$

⋆ LTI sistemde çıkışın frekansı giriş ile aynıdır

LTI $\Rightarrow T_{0,y} = T_{0,x} = \frac{2\pi}{3}$

b) $P_x = 3^2 + \frac{5^2}{2} = 21{,}5$ watt $\quad (\omega_0 = 3 = 2\pi f_0 \Rightarrow f_0 = \frac{3}{2\pi})$

$$x(t) = \underbrace{3}_{x_0} + \underbrace{\frac{5}{2}e^{j\pi/4}\, e^{j3t}}_{x_1} + \underbrace{\frac{5}{2}e^{-j\pi/4}\, e^{-j3t}}_{x_2} \qquad \omega_0$$

$$\rightarrow P_x = \sum_n |x_n|^2 = 3^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = 21{,}5 \text{ watt}$$

c) $Y(f) = X(f) \cdot H(f)$

$$H(f) = j20\pi f$$

$$X(f) = 3\delta(f) + \frac{5}{2}e^{j\pi/4}\,\delta(f-f_0) + \frac{5}{2}\,e^{-j\pi/4}\,\delta(f+f_0)$$

$$Y(f) = H(f)\cdot X(f) = (j20\pi f)\left[3\delta(f) + \frac{5}{2}e^{j\pi/4}\delta(f-f_0) + \frac{5}{2}e^{-j\pi/4}\delta(f+f_0)\right]$$

$$= \underbrace{3\delta(f)\cdot j20\pi(0)}_{0} + j20\pi\left(\frac{3}{2\pi}\right)\cdot\frac{5}{2}\,e^{j\pi/4}\,\delta(f-f_0)$$

$$+ j20\pi\left(-\frac{3}{2\pi}\right)\frac{5}{2}\,e^{-j\pi/4}\,\delta(f+f_0)$$

orj. s. 2

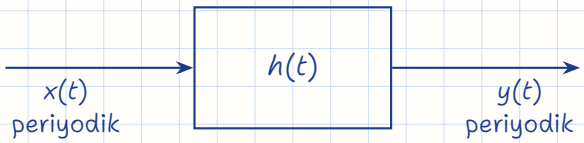
$$= 75\,\underbrace{j e^{j\pi/4}}_{e^{j3\pi/4}}\,\delta(f-f_0) - 75\,\underbrace{j e^{-j\pi/4}}_{e^{-j3\pi/4}}\,\delta(f+f_0)$$

$$Y(f) = 75e^{j3\pi/4}\,\delta(f-f_0) + 75e^{-j3\pi/4}\,\delta(f+f_0)$$

$\mathcal{F}^{-1}\downarrow$

$$y(t) = 150\cos\left(\overbrace{2\pi f_0}^{3}\,t + \frac{3\pi}{4}\right)$$

$$P_y = \frac{150^2}{2} \text{ watt} = 11250$$



$$x(t) = \sum_n x_n\,e^{j2\pi n f_0 t} \qquad y(t) = \sum_n y_n\,e^{j2\pi n f_0 t}$$

$$X(f) = \sum_n x_n\,\delta(f-nf_0) \qquad Y(f) = \sum_n y_n\,\delta(f-nf_0)$$

$$Y(f) = H(f)X(f) = H(f)\sum_n x_n\,\delta(f-nf_0)$$

$$= \sum_n x_n\,\underbrace{H(f)\,\delta(f-nf_0)}_{H(nf_0)\cdot\delta(f-nf_0)}$$

$$\boxed{Y(f) = \sum_n \underbrace{x_n H(nf_0)}_{y_n}\,\delta(f-nf_0)} \xrightarrow{\mathcal{F}^{-1}} y(t) = \sum_n y_n\,e^{j2\pi n f_0 t}$$

? Önch ex 2.2.6 inceleyiniz (57–59, 1. baskı)

Enerji ve Enerji yoğunluk fonksiyonu (EYF)

Energy Spectral Density (ESD)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} x(t)\cdot x^*(t)\,dt =$$

$$= \int_{-\infty}^{\infty} x(t)\Big[\underbrace{\int_{-\infty}^{+\infty} X(f)\,e^{j2\pi ft}\,df}_{=\int_{-\infty}^{\infty} X^*(f)\,e^{-j2\pi ft}}\Big]^*\,dt$$

orj. s. 3

$$= \int_{-\infty}^{\infty} X^*(f)\underbrace{\int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt}_{X(f)}\,df$$

$$= \int_{-\infty}^{\infty} |X(f)|^2\,df$$

Parseval t.:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}_{\mathcal{E}_x(f)\text{: EYF}}\,df$$

Enerji Tayf yoğunluğu F(EYF)

Enerji işaretleri için özilişki fonk (autocorrelation)

$$\underbrace{\mathcal{R}_x(\tau)}_{x(t)\text{'nin özilişki fonk}} = \int_{-\infty}^{+\infty} x(t)\cdot x^*(t-\tau)\,dt = \int_{-\infty}^{\infty} x(t)\,x^*(t+\tau)\,dt$$

$$= \int_{-\infty}^{+\infty} x^*(t)\,x(t+\tau)\,dt$$

$\mathcal{R}_x(-\tau) = \mathcal{R}_x(\tau)$ çift fonksiyondur

$\mathcal{R}_x(\tau) = x(\tau) * x(-\tau)$

$x(t)$ reel olsun $\Rightarrow$ $\mathcal{R}_x(\tau) = x(\tau) * x(-\tau)$

$\downarrow \mathcal{F}$

$$\mathcal{F}\{\mathcal{R}_x(\tau)\} = X(f)\cdot\frac{1}{|-1|}X\left(\frac{f}{-1}\right)$$

$$= X(f)\cdot\underbrace{X(-f)}_{X^*(f)}$$

$x(t)$ reelse $X(f) = X^*(-f)$

$$\mathcal{F}^{-1}\{\mathcal{R}_x(\tau)\} = |X(f)|^2 = \mathcal{E}_x(f)^1$$

$$\boxed{\mathcal{R}_x(\tau) \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{E}_x(f) = |X(f)|^2}$$

[1] Claude notu: $\mathcal{F}$ üzerindeki işaret okunaksız; formüle göre $\mathcal{F}\{\mathcal{R}_x(\tau)\}$ olmalı.

$$\mathcal{R}_x(\tau) = \underbrace{\int_{-\infty}^{\infty} x(t)\,x^*(t-\tau)\,dt}_{(\tau=0 \text{ olsun})} = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \mathcal{E}_x$$

Not: Enerji yoğunluk fonksiyonu max değeri $\tau = 0$'da olur[2]

$$\mathcal{E}_x = \mathcal{R}_x(0) = \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df = \int_{-\infty}^{+\infty} \mathcal{E}_x(f)\,df$$

orj. s. 4

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$
$$Y(f) = X(f) \cdot H(f)$$
$$|Y(f)|^2 = |X(f)|^2\,|H(f)|^2$$

$$\mathcal{E}_y(f) = \mathcal{E}_x(f)\,|H(f)|^2$$

$\mathcal{F}^{-1}$ ↓

$$\underbrace{\mathcal{F}^{-1}\{\mathcal{E}_y(f)\}}_{\mathcal{R}_y(\tau)} = \underbrace{\mathcal{F}^{-1}\{\mathcal{E}_x(f)\}}_{\mathcal{R}_x(\tau)} * \underbrace{\mathcal{F}^{-1}\{|H(f)^2|\}}_{\mathcal{R}_h(\tau)}$$

$$\mathcal{R}_y(\tau) = y(\tau) * y^*(-\tau) = \mathcal{R}_x(\tau) * \mathcal{R}_h(\tau)$$

Örnek

$x(t) = e^{-at}u(t)$, $a > 0$

a) $\mathcal{E}_x(f) = ?$ $\quad E_x = ?$ $\quad$ b) $\mathcal{R}_x(\tau) = ?$ $\quad \mathcal{E}_x(f) = ?$ $\quad E_x = ?$

a) $\mathcal{E}_x(f) = |X(f)|^2$

$$X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$$
$$= \int_0^{\infty} e^{-at} \underbrace{u(t)}_{\substack{=1 \; t\geq 0 \\ 0 \; t<0}} e^{-j2\pi ft}\,dt$$
$$= \int_0^{\infty} e^{-(a+j2\pi f)t}\,dt = \frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\Bigg|_0^{\infty}$$
$$= \frac{\overset{=0}{e^{-\infty}} - e^{-0}}{-(a+j2\pi f)} = \frac{1}{a+j2\pi f}$$

? okunamadı

$x = a + jb$

$|x| = \sqrt{a^2+b^2}$

$\theta_x = \arctan\left(\frac{b}{a}\right)$

$x = |x|\,e^{j\theta_x}$

$x^* = a - jb$

[2] Claude notu: $\tau = 0$ ifadesi nedeniyle muhtemelen özilişki fonksiyonu $\mathcal{R}_x(\tau)$ kastedilmiş.

$$X(f) = \frac{1}{a + j2\pi f} \Rightarrow \mathcal{E}_x(f) = |X(f)|^2 = \frac{1}{a^2 + (2\pi f)^2}$$

$$E_x = \int_{-\infty}^{\infty} \mathcal{E}_x(f)\,df = \int_{-\infty}^{\infty} \frac{df}{a^2 + (2\pi)^2 f^2}$$
$$= \frac{1}{2\pi a} \arctan\left(\frac{2\pi f}{a}\right)\Big|_{-\infty}^{\infty}$$
$$= \frac{1}{2\pi a}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \boxed{\frac{1}{2a}}$$

$$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \arctan\left(\frac{bx}{a}\right)$$

$$E_x = \frac{1}{2a}$$

orj. s. 5

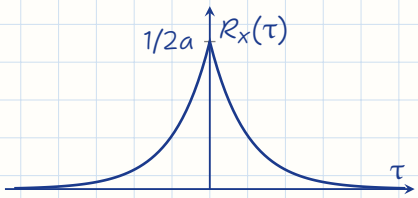
Sağlama:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} e^{-2at}\,u(t)\,dt$$
$$= \frac{e^{-2at}}{-2a}\Big|_0^{\infty} = -\frac{1}{2a}\left(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{-0}}_{1}\right)$$
$$= \frac{1}{2a}$$

b)

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\,x^*(t-\tau)\,dt$$
$$= \int_{-\infty}^{\infty} e^{-at} \underbrace{u(t)}_{=1 \;\; t\geq 0} \cdot e^{-a(t-\tau)} \underbrace{u(t-\tau)}_{=1 \;\; \substack{t-\tau\geq 0 \\ t\geq\tau}} dt$$
$$= e^{a\tau} \int_{-\infty}^{\infty} e^{-2at}\,dt =$$
$$\Rightarrow e^{a\tau} \int_{\tau}^{\infty} e^{-2at}\,dt \qquad \tau > 0 \text{ pozitifken}$$
$$= e^{a\tau}\,\frac{e^{-2at}}{-2a}\Big|_{\tau}^{\infty} = -\frac{1}{2a}\,e^{a\tau}\left[e^{-\infty} - e^{-2a\tau}\right]$$
$$= \frac{e^{-a\tau}}{2a}, \quad \tau > 0$$

çift fonksiyon olduğundan $R_x(\tau) = \dfrac{e^{-a|\tau|}}{2a} \quad \forall\tau$



$$E_x = R_x(0) = \frac{1}{2a}$$

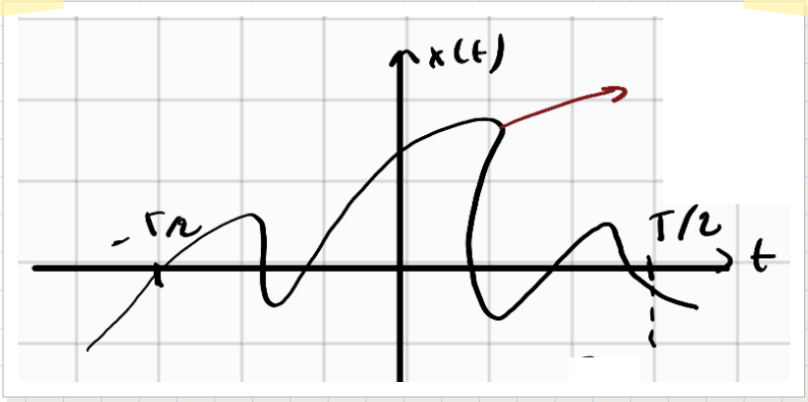
$$R_x(\tau) = \frac{e^{-a|\tau|}}{2a} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \mathcal{E}_x(f) = \frac{1}{a^2 + (2\pi f)^2}$$

$$e^{-a|\tau|},\ \tau > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{2a}{a^2 + (2\pi f)^2}$$ [3]

Güç ve Güç yoğunluk fonk (GYF)

orj. s. 6

Power Spectral Density (PSD)



orijinal çizim, s. 6

$$x_T(t) = \begin{cases} x(t), & |t| \le \dfrac{T}{2} \\ 0, & |t| > \dfrac{T}{2} \end{cases}$$

$$x(t) \text{ reel} \Rightarrow P_x = \lim_{T\to\infty} \frac{1}{T} \underbrace{\int_{-T/2}^{T/2} x^2(t)\,dt}_{E_{x_T}} = \lim_{T\to\infty} \frac{E_{x_T}}{T}$$

$$x_T(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_T(f)$$

$$E_{x_T} = \int_{-\infty}^{\infty} x_T^2(t)\,dt = \int_{-\infty}^{\infty} |X_T(f)|^2\,df$$

$$P_x = \lim_{T\to\infty} \frac{E_{x_T}}{T} = \lim_{T\to\infty} \frac{1}{T}\left[\int_{-\infty}^{\infty} |X_T(f)|^2\,df\right.$$

$T \nearrow \Rightarrow$ $x_T(t)$'nin süresi $\nearrow \Rightarrow E_{x_T} \nearrow \Rightarrow |X_T(f)|^2 \nearrow$

[3] Claude notu: koşul muhtemelen $a > 0$ olmalı.

$$P_x = \int_{-\infty}^{+\infty} \underbrace{\lim_{T\to\infty} \frac{1}{T}|X_T(f)|^2}_{=S_x(f)\text{ Güç yoğunluk fonk}}\, df = \int_{-\infty}^{+\infty} S_x(f)\, df$$

EYF, GYF pozitif, reel, çift f.

$x(t)$ birimi volt $\rightarrow$ volt^2/Hz

Güç işaretlerinin özilişki fonk

orj. s. 7

$x(t)$ güç işareti $\Rightarrow$

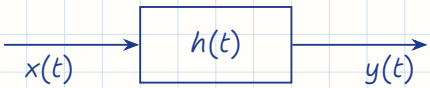
$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot x^*(t-\tau)\, dt$$

$$R_x(-\tau) = R_x(\tau)$$

$$R_x(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$

$$R_x(0) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = P_x$$

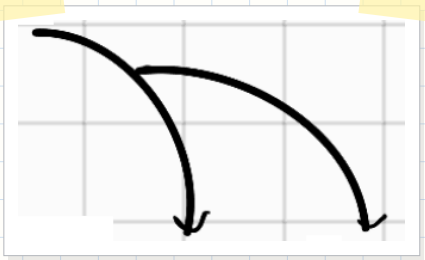
$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df$$



güç işareti:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

güç işareti: $R_y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} y(t)\, y^*(t-\tau)\, dt$



orijinal çizim, s. 7

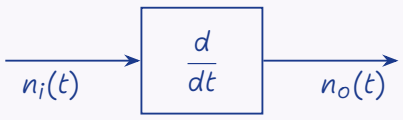
$$R_y(\tau) = R_x(\tau) * \underbrace{h(\tau) * h^*(-\tau)}_{R_h(\tau)}$$

$$\downarrow \mathcal{F}$$

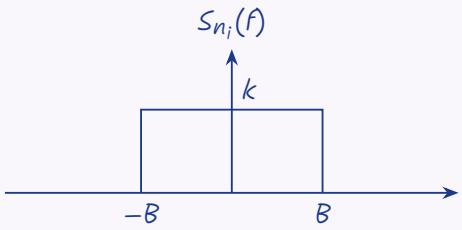
$$S_y(f) = S_x(f)\cdot H(f)\cdot H^*(f)$$

$$S_y(f) = S_x(f)\,|H(f)|^2$$

Örnek



$H(f) = j2\pi f$



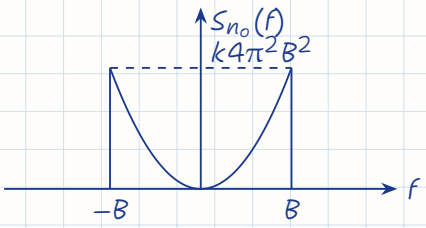
$n_i(t)$ is noise signal with PSD

$$S_{n_i}(f) = \begin{cases} k, & |f| \le B \\ 0, & \text{else} \end{cases}$$

$S_{n_o}(f) = ?$ $\quad P_{n_o} = ?$

orj. s. 8

$$S_{n_o}(f) = S_{n_i}(f)\,|H(f)|^2$$
$$= \begin{cases} k \cdot 4\pi^2 f^2, & |f| \le B \\ 0, & \text{Else} \end{cases}$$



$$P_{n_o} = \int_{-\infty}^{\infty} S_{n_o}(f)\,df = \int_{-B}^{B} k\,4\pi^2 B^2 f^2\,df^4$$

$$= \frac{8k\pi^2 B^3}{3}$$

Örnek

$x(t)$ periyodik $\Rightarrow$ $R_x(\tau) = ?$ $\quad S_x(f) = ?$

$\Downarrow$ (güç)

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cdot x^*(t-\tau)\,dt$$

[4] Claude notu: integrandda B^2 çarpanı muhtemelen fazla ($S_{n_o}(f) = k\cdot 4\pi^2 f^2$); sonuç $8k\pi^2B^3/3$ ile uyumlu olması için integrand $k\,4\pi^2 f^2$ olmalı.

$x(t)$ periyodik $\Rightarrow$ $x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t}$

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{\sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t}}_{x(t)} \cdot \underbrace{\sum_{m=-\infty}^{+\infty} x_m^* e^{-j2\pi m f_0 (t-\tau)}}_{x^*(t)} dt$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{\infty} x_n x_m^* e^{j2\pi m f_0 \tau} \underbrace{\int_{-T_0/2}^{T_0/2} e^{j2\pi n f_0 t}\cdot e^{-j2\pi m f_0 t} dt}_{\begin{cases} T_0, & n = m \\ 0, & n \neq m \end{cases}}$$

$$\boxed{R_x(\tau) = \sum_{n=-\infty}^{+\infty} |x_n|^2 e^{j2\pi n f_0 \tau}} \longleftrightarrow S_x(f) = \sum_{n=-\infty}^{+\infty} |x_n^2|\,\delta(f - nf_0)$$

Periyodik işaretin öz fonksiyonu periyodiktir

$$\rightarrow P_x = \int_{-\infty}^{+\infty} S_x(f)\,df = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} |x_n|^2\,\delta(f - nf_0)\,df$$

$$= \sum_n |x_n|^2 \underbrace{\int_{-\infty}^{+\infty} \delta(f - nf_0)\,df}_{1} \qquad P_x = \sum_{n=-\infty}^{\infty} |x_n|^2$$