Analog Haberleşme

Uygulama 1: Systems, Fourier Series & Transform

Original notes: Analog-Uygulama-1.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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orig. p. 1

Example

Classify these system into linear and non linear

a) $y(t) = 2x(t) - 3$
b) $y(t) = e^{-t}x(t)$

Linearity Cond:

$$\left.\begin{aligned} x_1(t) &\xrightarrow{T[\cdot]} y_1(t) \\ x_2(t) &\xrightarrow{T[\cdot]} y_2(t) \end{aligned}\right\} \quad T[ax_1(t) + bx_2(t)] \overset{?}{=} a\underbrace{T[x_1(t)]}_{y_1(t)} + b\underbrace{T[x_2(t)]}_{y_2(t)}$$

a)

$$\begin{aligned} x_1(t) &\to y_1(t) = 2x_1(t) - 3 \\ x_2(t) &\to y_2(t) = 2x_2(t) - 3 \\ x_3(t) &= ax_1(t) + bx_2(t) \\ &\to y_3(t) = 2x_3(t) - 3 = 2ax_1(t) + 2bx_2(t) - 3 \\ &\overset{?}{\neq} a(2x_1(t) - 3) + b(2x_2(t) - 3) \end{aligned}$$

non lineer!

b) $y(t) = e^{-t}x(t)$

$$\begin{aligned} x_1(t) &\to y_1(t) = e^{-t}x_1(t) \\ x_2(t) &\to y_2(t) = e^{-t}x_2(t) \\ x_3(t) &= ax_1(t) + bx_2(t) \\ &\to y_3(t) = e^{-t}x_3(t) = e^{-t}ax_1(t) + e^{-t}bx_2(t) \\ &\overset{?}{=} a\underbrace{y_1(t)}_{e^{-t}x_1(t)} + b\underbrace{y_2(t)}_{e^{-t}x_2(t)} \quad \checkmark \end{aligned}$$

lineer system

Example

Determine whether the following systems are time variant or time invariant

a) $y(t) = 2x(t) + 3$

b) $y(t) = (t+2)x(t)$

Time invariant cond:

$$x(t) \to y(t), \qquad x(t-t_0) \to y(t-t_0)$$

a)

$$\left.\begin{array}{l} x_1(t) \xrightarrow{2x_1(t)+3} y_1(t) \\ x_2(t) = x(t-t_0) \end{array}\right\} \quad y_2(t) \stackrel{?}{=} y_1(t-t_0)$$

$$2x_2(t) + 3 = 2x(t-t_0) + 3 \stackrel{?}{=} 2x_1(t-t_0) + 3 \quad \checkmark$$

zamanda değişmez

orig. p. 2

b)

$$x_1(t) \to y_1(t) = (t+2)x_1(t)$$
$$x_2(t) = x_1(t-t_0) \to y_2(t) = (t+2)x_2(t)$$
$$y_2(t) = (t+2)x_1(t-t_0)$$
$$y_2(t) \stackrel{?}{=} y_1(t-t_0)$$
$$(t+2)x_1(t-t_0) \stackrel{?}{\neq} (t-t_0+2)x_1(t-t_0)$$

time variant

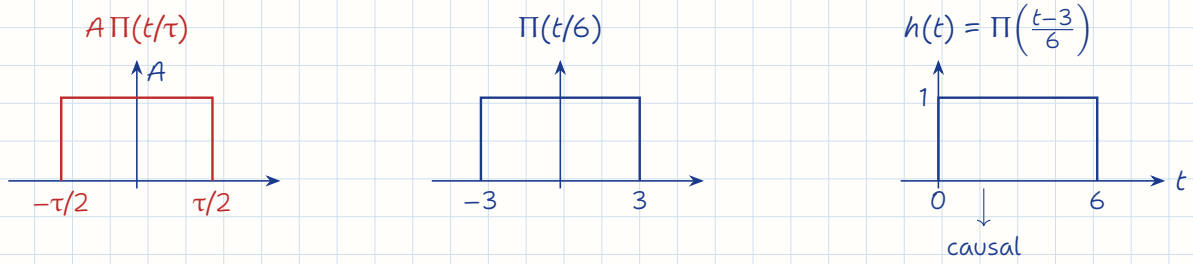
Example

A system is causal if its input at any time t_0 depends on the input at times prior to t_0

$$y(t_0) = T[x(t) : t \le t_0]$$

A nessesary an sufficent condition for an LTI system to be causal is that its impulse response $h(t)$, [the output when the input is $\delta(t)$] must be a causal siggnal for $t < 0$, we must have $h(t) = 0$

$h(t) = \Pi\left(\dfrac{t-3}{6}\right)$, LTI system is causal?



Example

Determine the Fourier Series Expansion of the following signals.

orig. p. 3

a) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-n)\,u(t-n)$

b) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-nT)$

Fourier Seri Açılımları

Trigonometrik FS:

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt, \qquad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

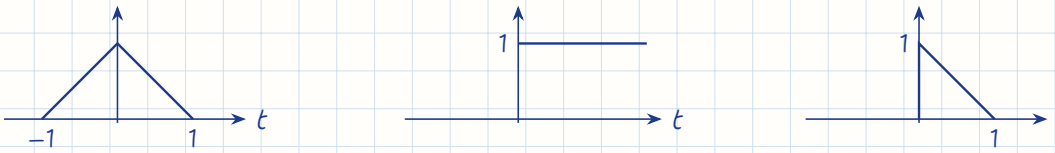
kompleks ExpFS:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}, \qquad x_n = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jn\omega_0 t}\,dt$$

$$x_n = \frac{a_n - jb_n}{2}$$ ← ikisi arasındaki ? bağıntı

a) önce $n = 0$ ve $n = 1$ için çizelim

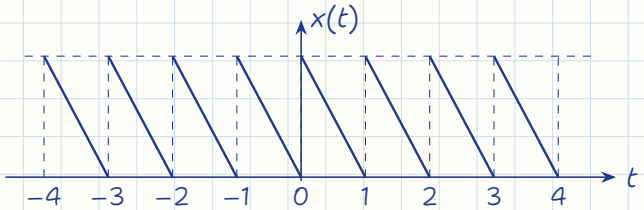
$$n = 0 \;\rightarrow\; \Lambda(t)\,u(t)$$



$$n = 1 \;\rightarrow\; \Lambda(t-1)\,u(t-1)$$



$$\sum_{n=-\infty}^{\infty} \Lambda(t-n)\,u(t-n)$$



orig. p. 4

kompleks FS

$$x_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$= \frac{1}{T_0}\int x(t)\,dt$$

$$= \int_0^1 (1-t)\,dt = \frac{1}{2}$$

$T_0 = 1$

$$y - y_0 = \overset{\text{doğrunun eğimi}}{\underset{-1}{m}}(x - x_0)$$

$$y - 1 = (-1)x$$

$$y(t) = 1 - t$$

$T_0 = 1$

$$n \neq 0 \rightarrow x_n = \int_0^1 x(t)\, e^{-jn2\pi t} = \int_0^1 (1-t)\, e^{-jn2\pi t}\, dt = \int_0^1 e^{-jn2\pi t}\, dt \underbrace{- \int_0^1 t\, e^{-jn2\pi t}\, dt}_{I_1}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\downarrow \quad -\frac{1}{jn2\pi} e^{-jn2\pi t}\Big|_0^1 = \frac{-1}{jn2\pi}\left[e^{-jn2\pi} - e^0\right]$$

$$I_1 = \int_0^1 t\, e^{-jn2\pi t}\, dt$$

$$\boxed{uv|_a^b - \int_a^b v\, du} \quad \leftarrow \textit{kısmi integrasyon}$$

$$u = t \qquad e^{-jn2\pi t}\, dt = dv$$

$$du = dt \qquad \frac{-1}{jn2\pi} e^{-jn2\pi t} = v$$

$$= \left(t \cdot \frac{-1}{jn2\pi} e^{-jn2\pi t}\right)\Big|_0^1 - \int_0^1 -\frac{1}{jn2\pi} e^{-jn2\pi t}\, dt$$

$$= -\frac{1}{jn2\pi} e^{-jn2\pi} + \frac{1}{n^2 4\pi^2} e^{-jn2\pi} - \frac{1}{n^2 4\pi^2}$$

$$x_n = \frac{1}{jn2\pi} + \frac{1}{n^2 4\pi^2} - \frac{1}{n^2 4\pi^2} e^{-jn2\pi}$$

$$= \frac{1}{jn2\pi} + \frac{1}{n^2 4\pi^2}\left[1 - \underbrace{e^{-jn2\pi}}_{1}\right] = \frac{1}{jn2\pi}$$

Üstel FS: $\boxed{x(t) = \frac{1}{2} + \sum_{n\neq 0} \frac{1}{jn2\pi} e^{jn2\pi t}}$

orig. p. 5

$$x_n = \frac{a_n - jb_n}{2} = \overset{j \text{ ile çarpalım}}{\frac{1}{jn2\pi}} = -\frac{j}{n2\pi}$$

$$a_n = 0, \qquad b_n = \frac{1}{n\pi}$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin(2\pi n t)$$

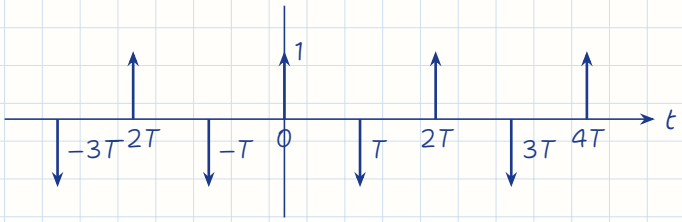
b)

$$n = 0 \rightarrow \delta(t)$$

$$n = 1 \rightarrow -\delta(t - T)$$



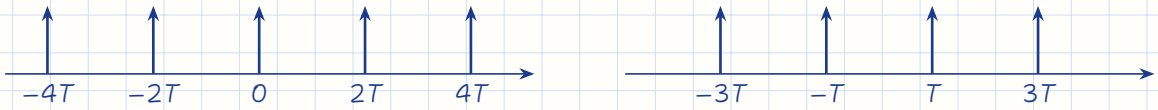
c)



Dürtü katarının FS açılımı

$$x(t) = \sum_n \delta(t - nT_0) = \frac{1}{T_0} \sum_n e^{jn\omega_0 t}$$

$$x(t) = \underbrace{\sum_n \delta(t - n2T)}_{} - \underbrace{\sum_n \delta(t - T - n2T)}_{}$$



$$x(t) = \frac{1}{2T} \sum_n e^{jn\frac{2\pi}{2T}t} - \frac{1}{2T} \sum_n e^{jn\frac{2\pi}{2T}(t-T)}$$

$$= \frac{1}{2T} \sum_n \left\{ e^{jn\frac{\pi}{T}t} - e^{jn\frac{\pi}{T}t} \underbrace{e^{-jn\frac{\pi}{T}\cdot T}}_{(-1)^n} \right\}$$

$$e^{-jn\pi} = \cos(-n\pi) + j\sin(-n\pi) = (-1)^n$$

$$= \frac{1}{2T} \sum_n e^{jn\frac{\pi}{T}\cdot t} \left(1 - (-1)^n\right)$$

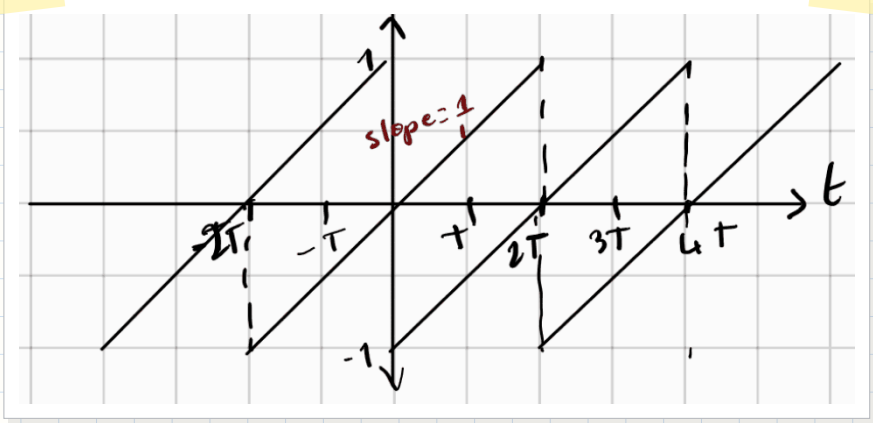
$$x_n = \frac{1}{2T}\left(1 - (-1)^n\right) = \begin{cases} \frac{1}{T} & n \text{ tek} \\ 0 & n \text{ çift} \end{cases}$$

orig. p. 6

$$\Rightarrow x(t) = \sum_n x_n e^{jn\frac{\pi}{T}\cdot t}, \qquad x_n = \begin{cases} 1/T, & n \text{ tek} \\ 0, & n \text{ çift} \end{cases}$$

Example

Determine the FS expansion of the following periodic signal



original drawing, p. 6

$$y = \frac{1}{T}x, \qquad x(t) = \frac{t}{T}$$

$$x_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$T_0 = 2T \;\rightarrow\; \omega_0 = \frac{2\pi}{2T} = \frac{\pi}{T}$$

$$x_n = \frac{1}{2T}\int_{-T}^{T} \frac{t}{T} e^{-jn\frac{\pi}{T}}\, dt$$

$$= \frac{1}{2T^2}\int_{-T}^{T} e^{-jn\frac{\pi t}{T}}\, dt^1$$

Claude notu: birinci satırda üstelde, ikinci satırda ise integrandda t çarpanı yazılmamış; sonraki adımlara göre ($u = t$) muhtemelen $x_n = \frac{1}{2T^2}\int_{-T}^{T} t\, e^{-jn\frac{\pi t}{T}}\, dt$ olmalı.

$$u = t \qquad e^{-jn\frac{\pi t}{T}}\, dt = dv$$

$$du = dt \qquad -\frac{1}{jn\frac{\pi}{T}} e^{-jn\frac{\pi t}{T}} = v$$

$$= \frac{1}{2T^2}\left[\left(t \cdot \frac{-1}{jn\frac{\pi}{T}} e^{-jn\frac{\pi t}{T}}\right)\Bigg|_{-T}^{T} - \int_{-T}^{T} -\frac{1}{jn\frac{\pi}{T}} e^{-jn\frac{\pi t}{T}}\, dt\right]$$

Sonuca giderken $e^{jn\pi} = (-1)^n$

x_0'ı ayrı olarak bul ödev

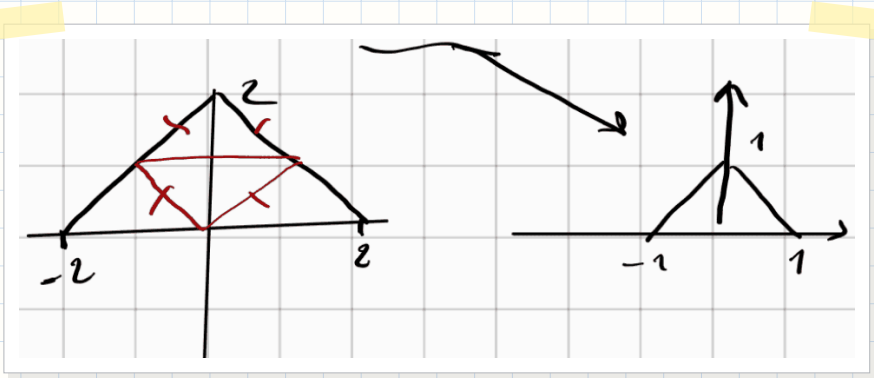
orig. p. 7

Example

Classify the following signal into periodic or non periodic

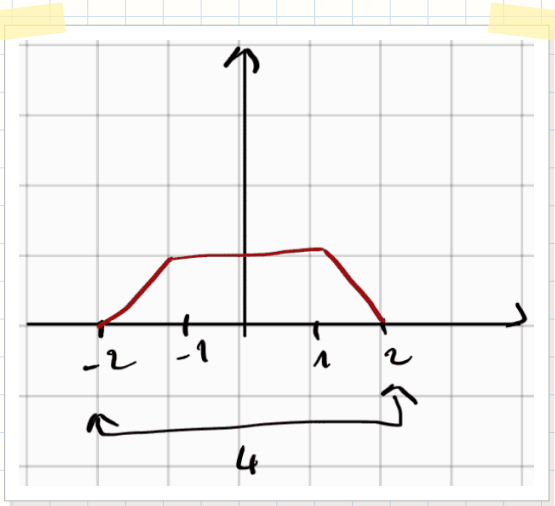
$$x(t) = 2\sum_n \Lambda\left(\frac{t-4n}{2}\right) - \sum_n \Lambda(t-4n)$$

$$n = 0 \;\rightarrow\; x_0(t) = x_1(t) = 2\Lambda\left(\frac{t}{2}\right) - \Lambda(t)$$

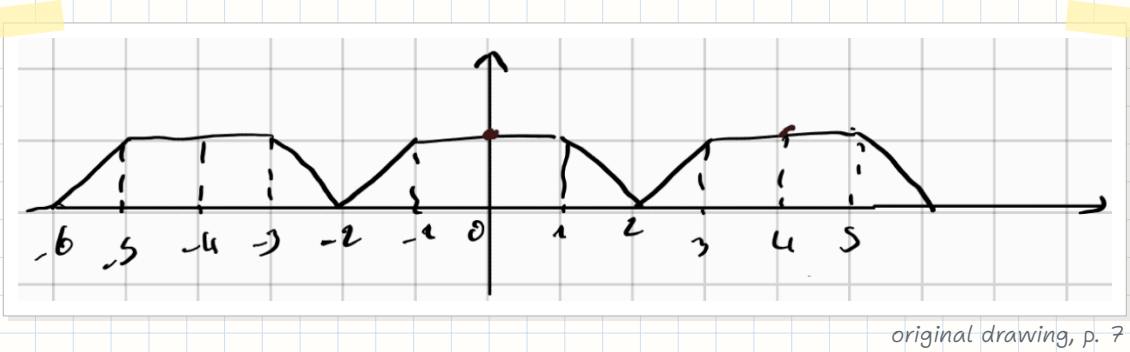


original drawing, p. 7

bunların farkı



original drawing, p. 7

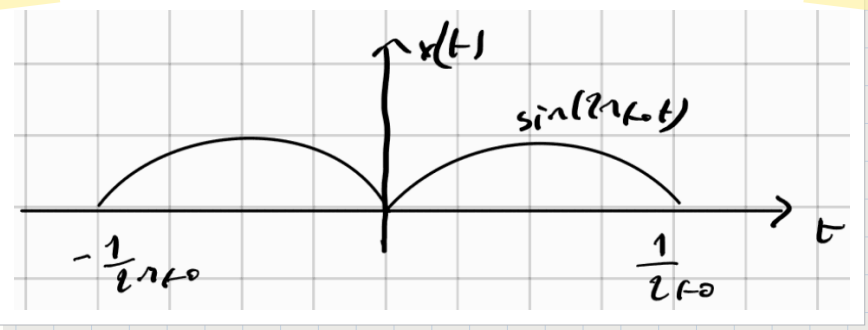


original drawing, p. 7

$T_0 = 4$ ile periyodiktir

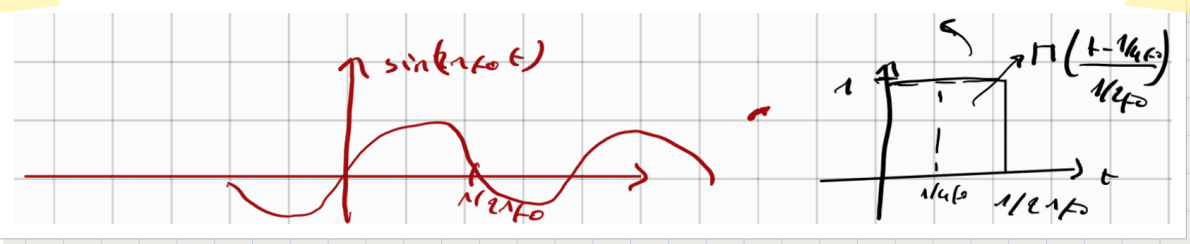
Example

Find the Fourier Transform of the following signal



original drawing, p. 7

$$X(f) = ?$$



original drawing, p. 7

orig. p. 8

$$x(t) = \sin(2\pi f_0 t)\cdot \Pi\left(\frac{t-\frac{1}{4f_0}}{\frac{1}{2f_0}}\right) - \sin(2\pi f_0 t)\,\Pi\left(\frac{t+\frac{1}{4f_0}}{\frac{1}{2f_0}}\right)$$

$$x(t) = \underbrace{\sin(2\pi f_0 t)}_{x_1(t)}\underbrace{\left[\Pi\left(2f_0\left(t-\tfrac{1}{4f_0}\right)\right) - \Pi\left(2f_0\left(t+\tfrac{1}{4f_0}\right)\right)\right]}_{x_2(t)}$$

$$x(t) = x_1(t)\cdot x_2(t)$$

$$\downarrow$$

$$X(f) = X_1(f) \circledast X_2(f)$$

$$\sin(2\pi f_0 t) \xrightarrow{\mathcal{F}} -\frac{1}{2j}\left[\delta(f+f_0) - \delta(f-f_0)\right] \quad \leftarrow X_1(f)$$

$$A\,\Pi\left(\frac{t}{\tau}\right) \xrightarrow{\mathcal{F}} A\,\tau\,\mathrm{sinc}(f\tau)$$

$$x(t-t_0) \rightarrow X(f)\,e^{-j2\pi f t_0}$$

$$x_2(t) = \Pi\Big(\underbrace{2f_0}_{\tau=\frac{1}{2f_0}}\Big(t - \underbrace{\tfrac{1}{4f_0}}_{t_0=\frac{1}{4f_0}}\Big)\Big) - \Pi\Big(2f_0\Big(t + \underbrace{\tfrac{1}{4f_0}}_{t_0=-\frac{1}{4f_0}}\Big)\Big)$$

$$X_2(f) = \frac{1}{2f_0}\,\mathrm{sinc}\left(f\cdot\frac{1}{2f_0}\right)e^{-j2\pi f\cdot\frac{1}{4f_0}} - \frac{1}{2f_0}\,\mathrm{sinc}\left(f\cdot\frac{1}{2f_0}\right)\cdot e^{j2\pi f\cdot\frac{1}{4f_0}}$$

$$= \frac{j}{f_0}\cdot\mathrm{sinc}\left(\frac{f}{2f_0}\right)\sin\left(\frac{\pi f}{2f_0}\right)$$

$$X(f) = X_2(f) \circledast \left[\frac{1}{2j}(\delta(f-f_0) - \delta(f+f_0))\right]$$

$$= \frac{1}{2j}\underbrace{X_2(f) \circledast \delta(f-f_0)}_{X_2(f-f_0)} - \frac{1}{2j}\underbrace{X_2(f) \circledast \delta(f+f_0)}_{X_2(f+f_0)}$$

$$x(t)\,\delta(t-t_0) = x(t_0)\cdot\delta(t-t_0)$$

$$x(t) \circledast \delta(t-t_0) = x(t-t_0)$$

orig. p. 9

Example

$$x(t) = 4\Lambda\left(\frac{t}{4}\right)\cos(2\pi f_0 t)$$

Find the Fourier Transform

Reminder:

$$A\Lambda(t/\tau) \rightarrow A\tau\,\mathrm{sinc}^2(f\tau)$$

$$\cos(2\pi f_0 t) \rightarrow \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$x(t)\cos(2\pi f_0 t) \rightarrow \frac{1}{2}[X(f-f_0) + X(f+f_0)]$$

Example

$$x(t) = \Lambda(2t+3) - \Lambda(3t-2)$$

Find the fourier transform

Reminder

$$A\Lambda(t/\tau) \rightarrow A\tau\,\mathrm{sinc}^2(f\tau)$$

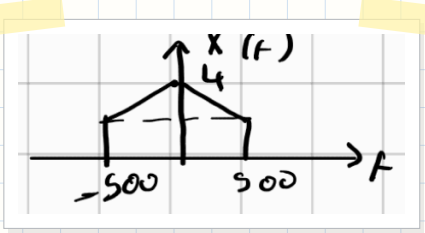
$$x(t-t_0) \rightarrow X(f)\,e^{-j2\pi f t_0}$$

$$x(t) = \underbrace{\Lambda\left(\frac{t+3/2}{1/2}\right)}_{\substack{A=1\\ \tau=1/2\\ t_0=-3/2}} - \underbrace{\Lambda\left(\frac{t-2/3}{1/3}\right)}_{\substack{A=1\\ \tau=1/3\\ t_0=2/3}}$$

$$X(f) = \frac{1}{2}\,\mathrm{sinc}^2\left(\frac{f}{2}\right)e^{j2\pi f\frac{3}{2}} - \frac{1}{3}\,\mathrm{sinc}^2\left(\frac{f}{3}\right)e^{-j2\pi f\frac{2}{3}}$$

Example

Fourier transform of a signal is



original drawing, p. 9

Determine and sketch the fourier transform of the signal

$$x_1(t) = \cancel{-x(t)} + x(t)\underbrace{\cos(2000\pi t)}_{2\pi f_0 t} + \overbrace{2x(t)}^{\uparrow x(t)}\underbrace{\cos^2(3000\pi t)}_{\frac{1}{2}[1+\cos(6000\pi t)]}$$

(Not: $-x(t)$ terimi kırmızı kalemle çizilerek iptal edilmiş.)

orig. p. 10

$$X_1(f) = \frac{1}{2}X(f-1000) + \frac{1}{2}X(f+1000) + \frac{1}{2}X(f-3000) + \frac{1}{2}X(f+3000)$$

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

Example

$$x(t) = t\,\mathrm{sinc}(t)$$

Find the fourier transform

1. yol

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \longrightarrow 2\pi f_0 t$$

$$x(t) = \frac{1}{\pi}\sin(\pi t) \qquad 2\pi f_0 t = \pi t$$

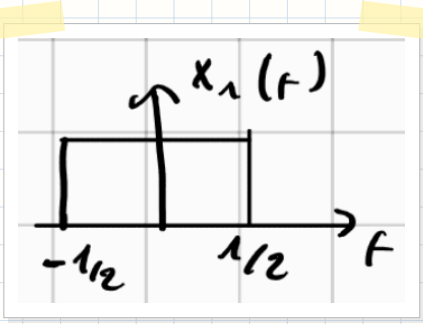
$$f_0 = ? = \frac{1}{2}$$

$$X(f) = \frac{1}{2j\pi}\delta\left(f-\frac{1}{2}\right) - \frac{1}{2j\pi}\delta\left(f+\frac{1}{2}\right)$$

2. yol Reminder:

$$t\,x(t) \rightarrow \frac{j}{2\pi}\frac{dX(f)}{df}$$

$$x_1(t) = \mathrm{sinc}(t) \rightarrow X_1(f) = \Pi(f)$$



original drawing, p. 10

$$X_1(f) = U\left(f+\frac{1}{2}\right) - U\left(f-\frac{1}{2}\right)$$

$$\frac{dX_1(f)}{df} = \delta\left(f+\frac{1}{2}\right) - \delta\left(f-\frac{1}{2}\right)$$

$$X(f) = \frac{j}{2\pi}\left[\delta\left(f+\frac{1}{2}\right) - \delta\left(f-\frac{1}{2}\right)\right]$$

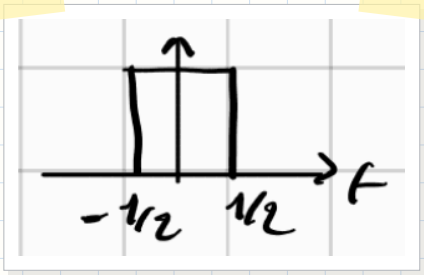
Example

$$x(t) = \mathrm{sinc}(t) \circledast \mathrm{sinc}(t) = \mathrm{sinc}(t)$$

Prove.

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(f)\cdot X_2(f)$$

$$x_1(t) = \text{sinc}(t) \longrightarrow X_1(f) = \Pi(f)$$
$$x_2(t) = \text{sinc}(t) \longrightarrow X_2(f) = \Pi(f)$$
$$X(f) = \Pi(f)$$
$$\downarrow \mathcal{F}^{-1}$$
$$x(t) = \text{sinc}(t)$$



original drawing, p. 10

orig. p. 11

Example

$$x(t) = \frac{1}{1+t^2} \qquad X(f) = ?$$

Reminder:

$$e^{-\alpha|t|} \rightarrow \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \rightarrow \frac{2\alpha}{4\pi^2}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + f2}\right]$$

Dualite

$$x(t) \rightarrow X(f)$$
$$X(t) \rightarrow x(-f)$$

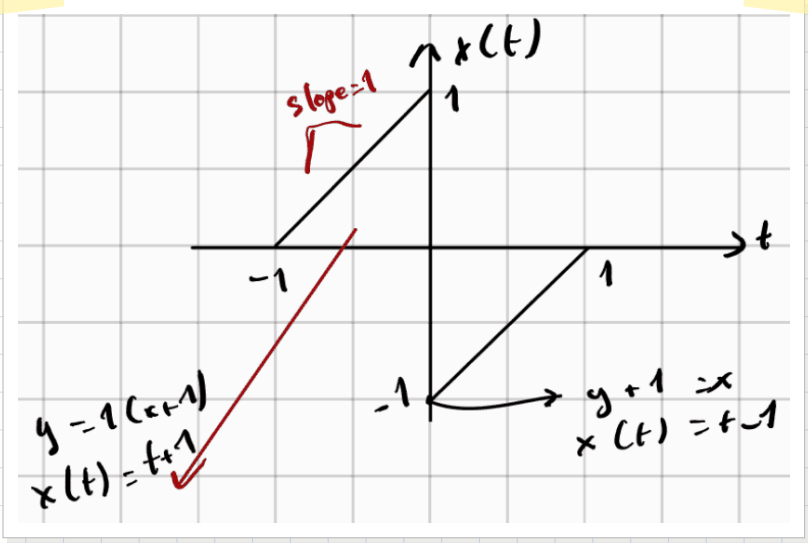
$$\frac{\alpha^2}{4\pi^2} = 1 \rightarrow \alpha^2 = 4\pi^2 \Rightarrow \alpha = 2\pi$$

$$X(t) = \frac{2 \cdot 2\pi}{4\pi^2}\left[\frac{1}{1+t^2}\right] = \frac{1}{\pi}\frac{1}{1+t^2}$$

$$x(-f) = e^{-\alpha|-f|} = e^{-\alpha|f|} = \pi \cdot e^{-\alpha|f|} = \pi \cdot e^{-2\pi|f|}$$

Example

$$X(f) = ? = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$



original drawing, p. 11

$$X(f) = \int_{-1}^{0} (t+1)\, e^{-j2\pi ft}\, dt + \int_{0}^{1} (t-1)\, e^{-j2\pi ft}\, dt$$

$$\text{sonuç} = \frac{j}{\pi f}\left[1 - \operatorname{sinc}(2f)\right]$$

$$= \underbrace{\int_{-1}^{0} t\, e^{-j2\pi ft}\, dt}_{I_1} + \underbrace{\int_{-1}^{0} e^{-j2\pi ft}\, dt}_{\frac{-1}{j2\pi f} e^{-j2\pi ft}\Big|_{-1}^{0}}$$

$$+ \underbrace{\int_{0}^{1} t\, e^{-j2\pi ft}\, dt}_{I_2} - \underbrace{\int_{0}^{1} e^{-j2\pi ft}\, dt}_{\frac{-1}{j2\pi f} e^{-j2\pi ft}\Big|_{0}^{1}}$$

orig. p. 12

kısmi integrasyon

$$\int_a^b u\, dv = u\, v\Big|_a^b - \int_a^b v\, du$$

$$I_1 = \int_{-1}^{0} t\, e^{-j2\pi ft}\, dt =$$

$u = t \qquad e^{-j2\pi ft}\, dt = dv$

$du = dt \quad = \dfrac{-1}{j2\pi f} e^{-j2\pi ft} = v$

$$= \left(t \frac{-1}{j2\pi f}\right.$$