İşaretler ve Sistemler

Ders 11 – Ayrık Zamanlı Fourier Dönüşümü (DTFT)

Orijinal not: Isaret-ve-Sistemler-ders-11.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Ayrık Zamanlı Fourier Transformu (Dönüşümü)

orj. s. 1

(Discrete-time Fourier Transform)

FT, periyodik olmayan işaretler için tanımlıdır.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega}), \qquad \mathcal{F}^{-1}$$

$\Rightarrow$ Ayrık zamanlı FT, $X(e^{j\omega})$ 2π ile periyodiktir.

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

Örnek

$x[n] = a^n u[n]$, $|a| < 1$ ise $X(e^{j\omega}) = ?$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \underset{u[n]=1,\ n\geq 0}{=} \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0} (ae^{-j\omega})^n \underset{\sum_{k=0}^{\infty}\alpha^k = \frac{1}{1-\alpha},\ |\alpha|<1}{=} \frac{1}{1-ae^{-j\omega}}$$

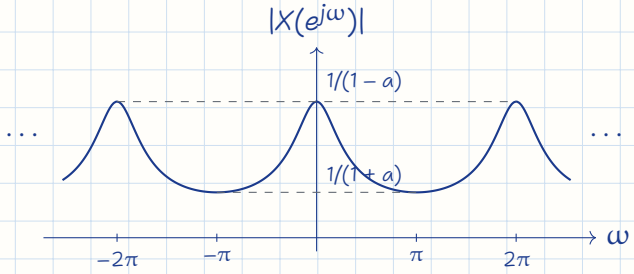
$$a^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-ae^{-j\omega}}$$

$$X(e^{j\omega}) = |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$$

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} \underset{e^{-j\omega}=\cos\omega - j\sin\omega}{=} \frac{1}{1-a\cos\omega + ja\sin\omega}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + a^2\sin^2\omega}} = \frac{1}{\sqrt{1+a^2-2a\cos\omega}}$$

$$\angle X(e^{j\omega}) = -\operatorname{atan}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$



Örnek

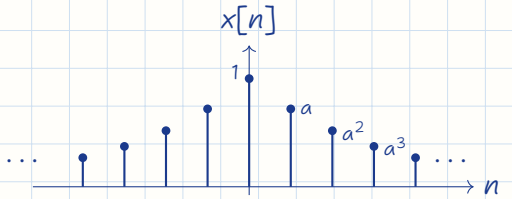
orj. s. 2

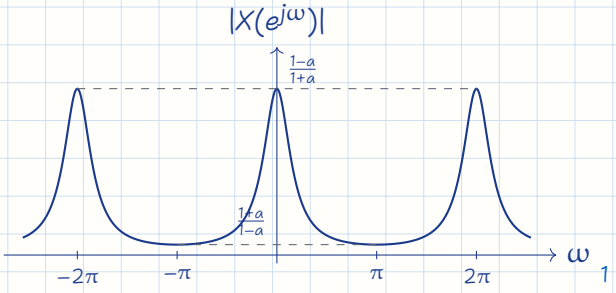
$x[n] = a^{|n|}$, $|a| < 1$ ise $X(e^{j\omega}) = ?$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n}e^{-j\omega n}$$

$$\sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n} \underset{k=-n}{=} \sum_{k=1}^{\infty} (ae^{j\omega})^{k} \underset{(ae^{j\omega})^0=1}{=} \sum_{k=0}^{\infty} (ae^{j\omega})^{k} - 1 = \frac{1}{1-ae^{+j\omega}} - 1 = \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$X(e^{j\omega}) = \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1+a^2-2a\cos\omega} : \text{Reel} \qquad X(e^{j\omega}) = |X(e^{j\omega})|, \quad \angle X(e^{j\omega}) = 0$$





[1]

$$a^{|n|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1-a^2}{1+a^2-2a\cos\omega}$$

Örnek

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$ şeklinde verilen ayrık zamanlı dikdörtgen darbe işaretinin FT bulunuz.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=0}^{2N_1} e^{-j\omega(n-N_1)} = e^{j\omega N_1}\sum_{n=0}^{2N_1} e^{-j\omega n}$$

$$= e^{j\omega N_1}\left[\frac{1-e^{-j\omega(2N_1+1)}}{1-e^{-j\omega}}\right]$$

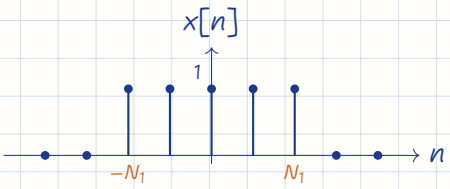
$$= e^{j\omega N_1}\frac{e^{-j\omega(N_1+1/2)}\left[e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

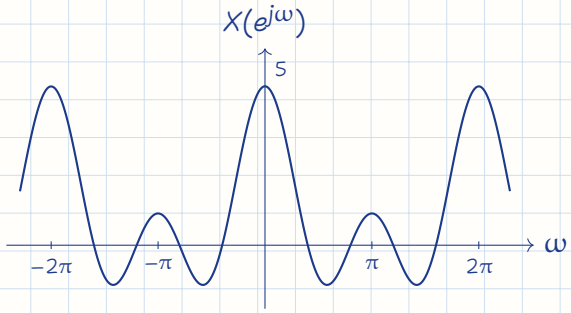
$$= \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$

[1] Claude notu: etiketler yer değiştirmiş görünüyor; muhtemelen tepe değeri ($\omega = 0$) $\frac{1+a}{1-a}$, en küçük değer ($\omega = \pm\pi$) $\frac{1-a}{1+a}$ olmalı.

Reminder: $\sum_{k=0}^{M} \alpha^k = \frac{1-\alpha^{M+1}}{1-\alpha}$, $\quad 1 - e^{j\theta} = e^{-j\theta/2}\left[e^{j\theta/2} - e^{-j\theta/2}\right]$[2]

$N_1 = 2$ için





Örnek

orj. s. 3

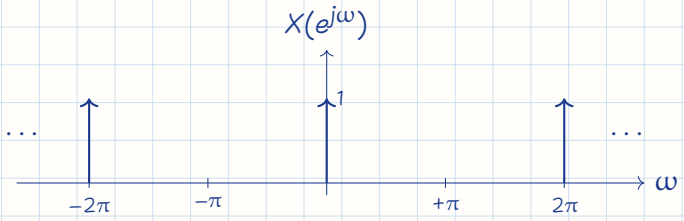
$x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \underset{\delta[n]=1,\ n=0}{=} e^{-j\omega 0} = 1$$

Örnek

$X(e^{j\omega}) = \delta(e^{j\omega})$ ise $x[n] = ?$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(e^{j\omega})\underbrace{e^{j0n}}_{1}\,d\omega = \frac{1}{2\pi}$$



$$\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1$$
$$1 \longleftrightarrow 2\pi\delta(e^{j\omega})$$

Örnek

$x[n] = \delta[n - n_0]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n-n_0]e^{-j\omega n} \underset{\delta[n-n_0]=1,\ n=n_0}{=} e^{-j\omega n_0}$$

$$\delta[n - n_0] \longleftrightarrow e^{-j\omega n_0}$$

Örnek

$X(e^{j\omega}) = \delta(e^{j(\omega-\omega_0)})$ ise $x[n] = ?$

[2] Claude notu: sağ taraf $1 - e^{-j\theta}$'ya eşittir; muhtemelen sol taraf $1 - e^{-j\theta}$ olmalı.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta\left(e^{j(\omega-\omega_0)}\right)e^{j\omega n}\,d\omega$$
$$= \frac{1}{2\pi}\int_{2\pi} \delta\left(e^{j(\omega-\omega_0)}\right)e^{j\omega_0 n}\,d\omega$$
$$= \frac{1}{2\pi}e^{j\omega_0 n}\underbrace{\int_{-\pi}^{\pi} \delta\left(e^{j(\omega-\omega_0)}\right)d\omega}_{=1} = \frac{1}{2\pi}e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta\left(e^{j(\omega-\omega_0)}\right)$$
$$e^{-j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta\left(e^{j(\omega+\omega_0)}\right)$$

Periyodik işaretlerin Fourier Transformu

orj. s. 4

$$x[n] = x[n+N],\ \forall n, \qquad N = \frac{2\pi}{\omega_0}, \qquad \omega_0 = \frac{2\pi}{N}$$

$$x[n] = \sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n} = x_0 + x_1 e^{j\omega_0 n} + x_2 e^{j2\omega_0 n} + \cdots + x_{N-1}e^{j(N-1)\omega_0 n}$$

$$x_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}, \qquad x_k = x_{k+N} \longleftarrow \begin{matrix} x_0 = x_N \\ x_1 = x_{N+1} \\ \vdots \end{matrix}$$

$$x[n] = \sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi x_k\,\delta(\omega - k\omega_0)$$

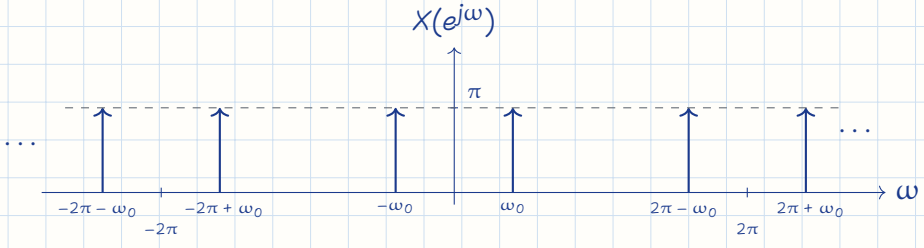
Örnek

$x[n] = \cos\left(\frac{2\pi n}{5}\right)$ ise $X(e^{j\omega}) = ?$

$$x[n] = \frac{1}{2}e^{j(2\pi n/5)} + \frac{1}{2}e^{-j(2\pi n/5)}, \qquad \omega_0 = 2\pi/5,\quad x_1 = x_{-1} = 1/2$$

$$\updownarrow \mathcal{F}$$

$$X(e^{j\omega}) = \pi\,\delta(\omega - 2\pi/5) + \pi\,\delta(\omega + 2\pi/5), \qquad -\pi \le \omega < \pi$$

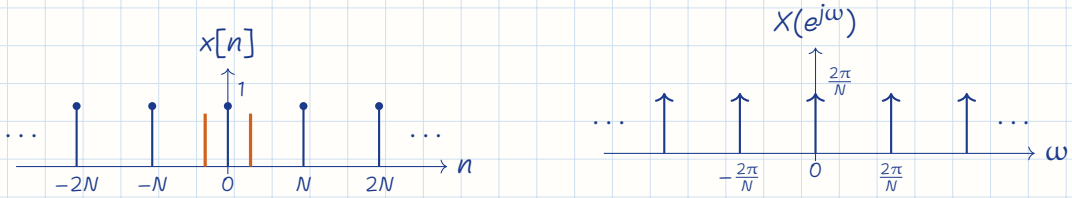


Örnek

$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$ periyodik dürtü treninin FT bulunuz.

Reminder: $x_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}$

$$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{N}\right)$$



Ayrık Zamanlı Fourier Transformunun Özellikleri

orj. s. 5

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$
$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$
$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

1) **Periyodiklik:** $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$, 2π ile periyodiktir.
2) **Lineerlik:** $\alpha x_1[n] + \beta x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$
3) **Zamanda öteleme:** $x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}X(e^{j\omega})$
   Genlik spektrumu değişmez, faz spektrumu $-\omega n_0$ kadar ötelenir.
4) **Frekansta öteleme:** $e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$
5) **Zamanda katlama:** $x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$
6) **Eşlenik alma:** $x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$
7) **Fark alma (differencing):** $x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1-e^{-j\omega})X(e^{j\omega})$
8) **Birikim (Accumulation):**

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$$

9) **Frekansta türev:** $n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$

   İspat: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{d\omega}(e^{-j\omega n}) = -jn\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$[3]

10) **Parseval teoremi:** $E_x = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2\,d\omega$
11) **Çarpma:** $x_1[n]\cdot x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\alpha})X_2(e^{j(\omega-\alpha)})\,d\alpha$

[3] Claude notu: n toplam indisidir, toplamın dışına çıkamaz; muhtemelen $= -j\sum_n n\,x[n]e^{-j\omega n}$ olmalı.

12) **Konvolüsyon:** $x_1[n] * x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})X_2(e^{j\omega})$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * \underset{\text{Dürtü cevabı}}{h[n]} \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})\ \underset{\text{Frekans cevabı}}{H(e^{j\omega})}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} h[n]$$

Örnek

orj. s. 6

$x[n] = u[n]$ ise $X(e^{j\omega}) = ?$

Reminder: $u[n] = \sum_{k=-\infty}^{n} \delta[k] = \sum_{k=0}^{\infty} \delta[n-k]$

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

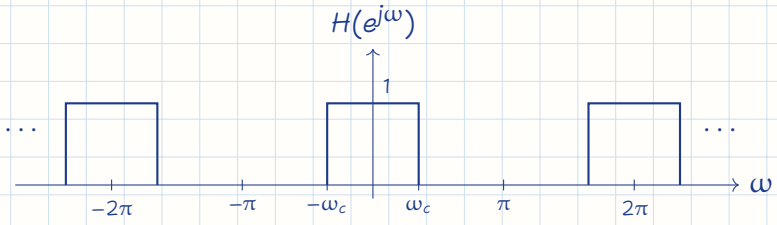
$$\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Örnek

$$H(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

ayrık zamanlı alçak geçiren filtrenin dürtü cevabını bulunuz.



$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{+j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\frac{e^{j\omega n}}{jn}\Big|_{-\omega_c}^{\omega_c} = \frac{1}{2j\pi n}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

$$h[n] = \frac{1}{\pi n}\sin(\omega_c n) = \frac{\sin\left(\pi\frac{\omega_c}{\pi}n\right)}{\pi\frac{\omega_c}{\pi}n}\cdot\frac{\omega_c}{\pi} = \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

