İşaretler ve Sistemler

Ders 12 – Laplace Transformu

Orijinal not: Isaret-ve-Sistemler-ders-12.pdf



UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Laplace Transformu

orj. s. 1

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(s) = \mathcal{L}\{x(t)\}$$

$$\Rightarrow s = \sigma + j\omega \begin{cases} \longrightarrow \sigma: \text{ Reel kısım} \\ \hookrightarrow \omega: \text{ Sanal kısım} \end{cases}$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

$\Rightarrow s = j\omega$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \mathcal{F}\{x(t)\} = X(s)\Big|_{s=j\omega}$$

Laplace transformu, $x(t)e^{-\sigma t}$ şeklinde $x(t)$ işaretinin bir reel üstel işaret ile çarpımının Fourier transformudur.

$$e^{-\sigma t} \begin{cases} \longrightarrow \sigma > 0 \text{ zamanla azalan fonksiyon} \\ \hookrightarrow \sigma < 0 \text{ zamanla artan fonksiyon} \end{cases}$$

Örnek

$x(t) = e^{-at}\underline{u(t)}$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(a+s)t}\,dt = \frac{e^{-(a+s)t}}{-(a+s)}\Big|_0^{\infty} = \underline{\frac{1}{s+a}},\quad \underline{\mathrm{Re}\{s+a\} > 0}$$

$$s = \sigma + j\omega \longrightarrow X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega},\quad \sigma + a > 0$$

$$s = j\omega,\ \sigma = 0 \longrightarrow X(j\omega) = \frac{1}{a + j\omega},\quad a > 0$$

$$\left.\begin{aligned} e^{-at}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a},\quad \mathrm{Re}\{s\} > \underline{-a} \\ u(t) &\overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s},\quad \mathrm{Re}\{s\} > 0 \end{aligned}\right\}\ a = 0$$

Örnek

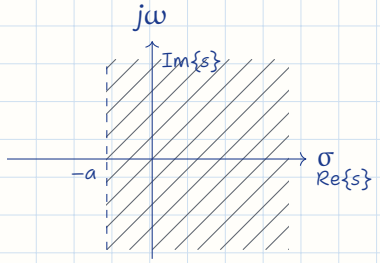
$x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}\,dt = -\int_{-\infty}^{0} e^{-(a+s)t}\,dt = \frac{1}{s+a}, \quad \mathrm{Re}\{s+a\} < 0$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

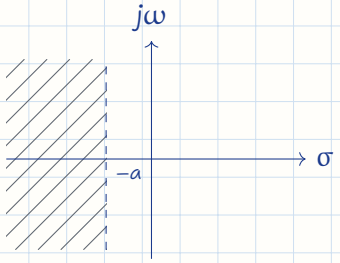
orj. s. 2

⇒ **Yakınsaklık bölgesi (Region of convergence, ROC):** Laplace transform integralinin yakınsadığı s değer aralığına yakınsaklık bölgesi denir.



s-düzlemi

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}, \quad \underline{\mathrm{Re}\{s\} > -a}$$



s-düzlemi

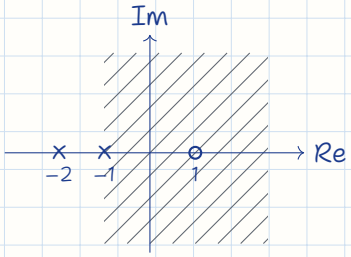
$$\mathcal{L}\{-e^{-at}\underline{u(-t)}\} = \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

Örnek

$x(t) = \underline{3e^{-2t}u(t) - 2e^{-t}u(t)}$

$$\left.\begin{aligned} e^{-2t}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2 \\ e^{-t}u(t) &\longleftrightarrow \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1 \end{aligned}\right)$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}, \quad \mathrm{Re}\{s\} > -1$$



s-düzlemi

x: pole (root of denumerator) (kutup)

o: zero (root of numerator) (sıfır)

Örnek

$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{2}e^{2t}u(t)$

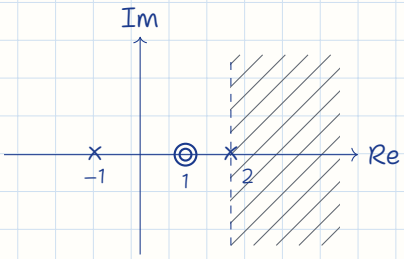
$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}\,dt = \int_{-\infty}^{\infty} \delta(t)e^{-s0}\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1, \quad \forall s \text{ (ROC, bütün s-düzlemi)}$$

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$$e^{2t}u(t) \longleftrightarrow \frac{1}{s-2}, \quad \mathrm{Re}\{s\} > 2$$

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-2} = \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathrm{Re}\{s\} > 2^1$$



⇒ Aynı noktada tekrar eden s sayısına mertebe denir.
Paydada, $s = -1$ ve $s = 2$ olmak üzere birinci mertebeden iki adet kutup ve payda $s = 1$'de ikinci mertebeden bir adet sıfır vardır.

Not: Laplace transformunun ROC'u $j\omega$ eksenini içermiyorsa Fourier transformu yakınsamaz.
$\frac{1}{3}e^{2t}u(t)$ ifadesinin Fourier transformu yoktur.

Yakınsama bölgesi (ROC) özellikleri

1) $X(s)$'in ROC'u s-düzlemindeki $j\omega$ eksenine paralel şeritlerden oluşur.

$$X(s) \underset{\substack{\uparrow \\ s=\sigma+j\omega}}{=} X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt = \mathcal{F}\{\underline{x(t)e^{-\sigma t}}\}$$

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}\,dt \underline{< \infty}$$

2) Rasyonel Laplace transform için, ROC herhangi bir kutup içermez.
3) $x(t)$ sonlu süreli ve mutlak entegre edilebilir ise, ROC bütün s düzlemidir.

$$\int_{T_1}^{T_2} |x(t)|\,dt < \infty$$
$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < \infty \longleftarrow \text{1) özellik}$$

→ $\mathrm{Re}\{s\} = \sigma = 0$ iken $s = j\omega$ olup ROC içinde kalır.
→ $\mathrm{Re}\{s\} = \sigma > 0$ iken $\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < e^{-\sigma T_1}\underbrace{\int_{T_1}^{T_2} |x(t)|\,dt}_{\infty} < \infty$ olur ve $\mathrm{Re}\{s\} > 0$ s-düzlemi ROC içinde kalır.
→ $\mathrm{Re}\{s\} = \sigma < 0$ iken $\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < e^{-\sigma T_2}\int_{T_1}^{T_2} |x(t)|\,dt < \infty$ olur ve $\mathrm{Re}\{s\} < 0$ s-düzlemi ROC içinde kalır.

Örnek

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{dışında} \end{cases}$$

[1] Claude notu: $\frac{1}{2}$ katsayısıyla toplam $\frac{s^2 - \frac{11}{6}s + \frac{7}{6}}{(s+1)(s-2)}$ olur; $\frac{(s-1)^2}{(s+1)(s-2)}$ sonucu katsayı $\frac{1}{3}$ iken elde edilir (muhtemelen $x(t)$'de $\frac{1}{3}e^{2t}u(t)$ olmalı).

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{0}^{T} e^{-(a+s)t}\,dt = \frac{1}{s+a}\left(1 - e^{-(a+s)T}\right) \longleftarrow$$

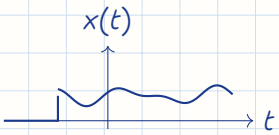
3)'e göre ROC bütün s düzlemi
2)'e göre ROC kutup içermez, $s = -a$ kutup var.

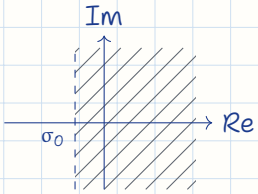
$s = -a$ iken $X(s) = \frac{0}{0}$

orj. s. 4

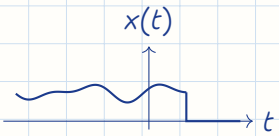
$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left(\frac{\frac{d}{ds}\left(1 - e^{-(a+s)T}\right)}{\frac{d}{ds}(s+a)}\right) = \lim_{s\to -a} Te^{-aT}e^{-sT} \Longrightarrow \underline{X(-a) = T}$$

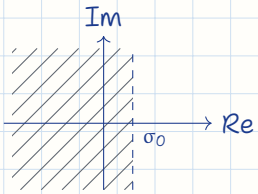
4) $x(t)$ sağ yönlü ve $\mathrm{Re}\{s\} = \sigma_0$ doğrusu ROC'da ise, $\underline{\mathrm{Re}\{s\} > \sigma_0}$ için bütün s değerleri de ROC'da olur.



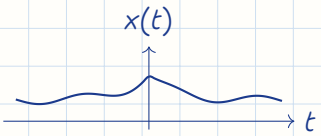


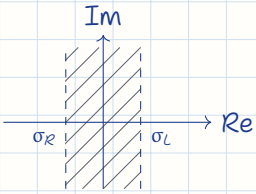
5) $x(t)$ sol yönlü ve $\mathrm{Re}\{s\} = \sigma_0$ doğrusu ROC'da ise, $\mathrm{Re}\{s\} < \sigma_0$ için bütün s değerleri de ROC'da olur.





6) $x(t)$ çift yönlü ve $\mathrm{Re}\{s\} = \sigma_0$ doğrusu ROC'da ise, ROC, s düzleminde $\mathrm{Re}\{s\} = \sigma_0$ doğrusunu içeren bir şeritten oluşur.





7) $X(s)$ rasyonel ise, ROC, ya kutuplarla sınırlıdır ya da sonsuza genişler ve kutup içermez.
→ $x(t)$ sağ yönlü ise, ROC, en sağdaki kutubun sağında kalan s düzlemidir.
→ $x(t)$ sol yönlü ise, ROC en soldaki kutubun solunda kalan s düzlemidir.

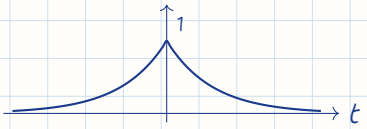
Örnek

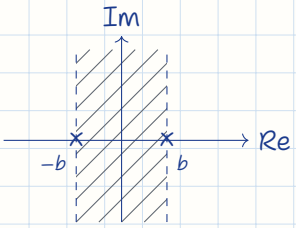
$x(t) = e^{-b|t|}$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{0} e^{bt}e^{-st}\,dt + \int_{0}^{\infty} e^{-bt}e^{-st}\,dt = \underline{\frac{-1}{s-b}} + \underline{\frac{1}{s+b}}, \qquad \begin{matrix}\mathrm{Re}\{s-b\} < 0 \\ \mathrm{Re}\{s+b\} > 0\end{matrix}$$

$$X(s) = \frac{-2b}{s^2 - b^2}, \qquad -b < \mathrm{Re}\{s\} < b$$

$x(t) = e^{-b|t|},\ b > 0$

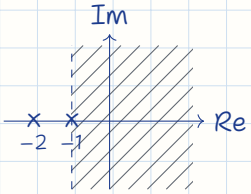




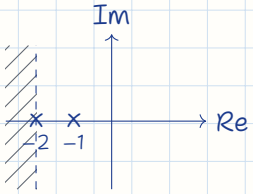
Örnek

orj. s. 5

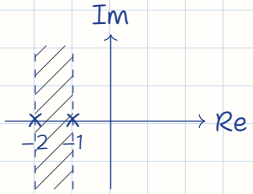
$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$



$x(t)$ sağ yönlü ise
$x(t) = e^{-t}u(t) - e^{-2t}u(t)$
ROC, $s = j\omega$ içerir, FT yakınsar.



$x(t)$ sol yönlü ise
$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$
FT yoktur



$x(t)$ çift yönlü ise
$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$
FT yoktur.

Ters Laplace Transform

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}\,d\omega$$

$$s = \sigma + j\omega \longrightarrow ds = j\,d\omega$$

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds}$$

$$\Rightarrow X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i} \quad \begin{cases} \longrightarrow x(t) = A_i e^{-a_i t}u(t) & : \text{sağ yönlü} \\ \hookrightarrow x(t) = -A_i e^{-a_i t}u(-t) & : \text{sol yönlü} \end{cases}$$

Laplace Transformunun Özellikleri

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s),\ \text{ROC}: R \qquad X(s) = \mathcal{L}\{x(t)\} \qquad x(t) = \mathcal{L}^{-1}\{X(s)\}$$

1) **Lineerlik:** $\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X_1(s) + \beta X_2(s)$, ROC: $R_1 \cap R_2$
2) **Zamanda öteleme:** $x(t - t_0) \longleftrightarrow e^{-st_0}X(s)$, ROC: R
3) **s-domeninde öteleme:** $e^{s_0 t}x(t) \longleftrightarrow X(s - s_0)$, ROC: $R + \text{Re}\{s_0\}$
   $s_0 = j\omega_0 \Rightarrow e^{j\omega_0 t}x(t) \longleftrightarrow X(s - j\omega_0)$, ROC: R
4) **Zamanda ölçeklendirme:** $x(at) \longleftrightarrow \frac{1}{|a|}X(s/a)$, ROC: R/a
   $a = -1 \Rightarrow x(-t) = X(-s)$, ROC: $-R$

5) **Eşlenik alma:** $x^*(t) \longleftrightarrow X^*(s^*)$, ROC: $\mathcal{R}$

6) **Türev:** $\frac{dx(t)}{dt} \longleftrightarrow sX(s)$, ROC: $\mathcal{R}$'yi içerir.

orj. s. 6

$$-tx(t) \longleftrightarrow \frac{dX(s)}{ds}, \quad \text{ROC: } \mathcal{R}$$

7) **Integral:** $\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{1}{s}X(s)$, ROC: $\mathcal{R} \cap \{Re\{s\} > 0\}$

8) **Konvolüsyon:** $x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s)$, ROC : $\mathcal{R}_1 \cap \mathcal{R}_2$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) \longleftrightarrow \begin{array}{l} Y(s) = X(s)H(s) \\ H(s) = \dfrac{Y(s)}{X(s)} \ : \text{ Transfer fonksiyonu} \end{array}$$

Örnek

$x(t) = te^{-at}u(t)$, $X(s) = ?$

$$te^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad Re\{s\} > -a$$

$$te^{-at}u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}, \quad Re\{s\} > -a$$

$$\frac{t^2}{2}e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^3}, \quad Re\{s\} > -a$$

$$\vdots$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^n}, \quad Re\{s\} > -a$$

Örnek

$H(s) = \dfrac{2s^2 + 5s + 5}{(s+1)^2(s+2)}$, $Re\{s\} > -1$, $h(t) = ?$

$$H(s) = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{(s+2)} \longrightarrow \begin{array}{l} A = 2 \\ B = -1 \\ C = 3 \end{array}$$

$$\updownarrow \mathcal{L}^{-1}$$

$$h(t) = 2te^{-t}u(t) - e^{-t}u(t) + 3e^{-2t}u(t)$$

ROC ile LTI sistemlerin özelliklerinin analizi

1) **Nedensellik:** $h(t) = 0$, $t < 0$ sağ yönlü $\longrightarrow$ ROC sağ yönlü

2) **Kararlılık:** $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \longrightarrow$ ROC, $j\omega$ eksenini içermelidir.

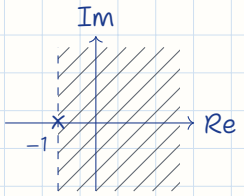
Örnek

orj. s. 7

$h(t) = e^{-t}u(t)$, kararlı mıdır?

$$\updownarrow$$

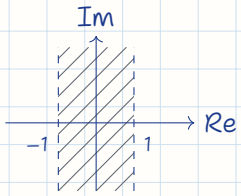
$$H(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$



ROC sağ yönlü olduğundan nedenseldir.

Örnek

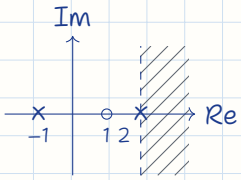
$$h(t) = e^{-|t|} \longleftrightarrow H(s) = \frac{-2}{s^2-1}, \quad -1 < \mathrm{Re}\{s\} < 1$$



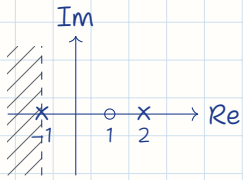
Nedensel değildir.

Örnek

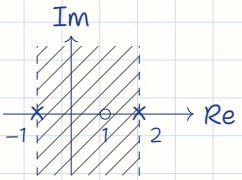
$$H(s) = \frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}, \qquad A = 2/3, \; B = \frac{1}{3}$$



$h(t)$ sağ yönlü
$h(t) = \frac{2}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$
Nedensel (sağ yönlü)
Kararlı değil (jω eksenini içermez)



$h(t)$ sol yönlü
$h(t) = -\frac{2}{3}e^{-t}u(-t) - \frac{1}{3}e^{2t}u(-t)$
Nedensel değil (sol yönlü)
Kararlı değil



$h(t)$ çift yönlü
$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$
Nedensel değil
Kararlı

Örnek

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t), \quad \underline{h(t) = ?}$$

$$\updownarrow \mathcal{L}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) + 3X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2}, \qquad \begin{matrix} A = 2 \\ B = -1 \end{matrix}$$

$$\updownarrow \mathcal{L}^{-1}$$

$$h(t) = 2e^{-t}u(t) - e^{-t}u(t), \quad \mathrm{Re}\{s\} > -1^{2}$$

[2] Claude notu: $B/(s+2)$ teriminin ters dönüşümü $e^{-2t}u(t)$'dir; muhtemelen $h(t) = 2e^{-t}u(t) - e^{-2t}u(t)$ olmalı.