İşaretler ve Sistemler

Ders 3 – Lineer zamanla değişmeyen (LZD / LTI) sistemler

Orijinal not: Isaret-ve-Sistemler-ders-3.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Lineer Zamanla Değişmeyen (LZD) Sistemler:

orj. s. 1

Linear Time-Invariant (LTI) Systems

LTI sistemler detaylı olarak analiz edilebilirler

- Üst üste ekleme (superposition) özelliği
- Dürtü cevabı (impulse response) ile tüm karakteristik özellikleri belirlenebilir.

Dürtü cevabı: Bir LTI sistemin dürtü cevabı $h(t)/h[n]$, sisteme birim dürtü işareti $\delta(t)/\delta[n]$ uygulandığında elde edilen çıkışıdır.

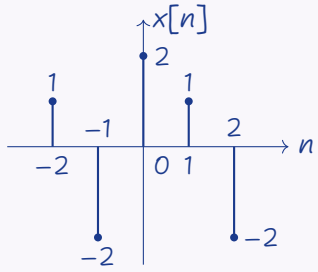
$$x(t) = \delta(t) \longrightarrow y(t) = h(t)$$
$$x[n] = \delta[n] \longrightarrow y[n] = h[n]$$

$\Rightarrow$ Herhangi bir $x[n]/x(t)$ işareti, zamanda kaydırılmış birim dürtü işaretlerinin lineer kombinasyonu ile ifade edilebilir.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = \cdots + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + \cdots$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Örnek (Ex)



Birim dürtü işareti cinsinden ifade ediniz.

$$x[n] = \sum_{k=-2}^{2} x[k]\,\delta[n-k] = x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n]$$
$$+ x[1]\,\delta[n-1] + x[2]\,\delta[n-2]$$
$$= \delta[n+2] - 2\,\delta[n+1] + 2\,\delta[n] + \delta[n-1] - 2\,\delta[n-2]$$

Ayrık zamanlı LTI sistemlerde konvolüsyon

orj. s. 2

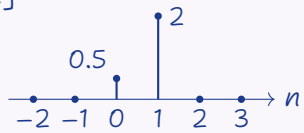


$$x[n] = \delta[n] \longrightarrow y[n] = h[n] \;:\; \text{Birim dürtü cevabı}$$
$$x[n] = \delta[n-1] \longrightarrow y[n] = h[n-1]$$
$$x[n] = 3\,\delta[n+2] \longrightarrow y[n] = 3\,h[n+2]$$
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \;:\; \text{Ayrık zamanlı konvolüsyon}$$

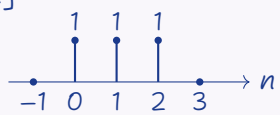
$$x[n] \longrightarrow \boxed{\overset{\text{LTI}}{h[n]}} \longrightarrow y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Örnek (Ex)

$x[n]$



$h[n]$

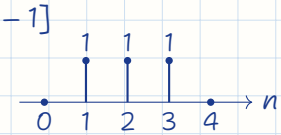


$y[n] = ?$

Çözüm:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \underset{k\in(0,1)}{=} x[0]\,h[n] + x[1]\,h[n-1] = \underbrace{0.5\,h[n] + 2\,h[n-1]}_{n\in(0,1,2)\quad n\in(1,2,3)}$$
$$n\in(0,1,2,3)$$

$h[n-1]$



$y[n] = 0, \quad n \le -1$

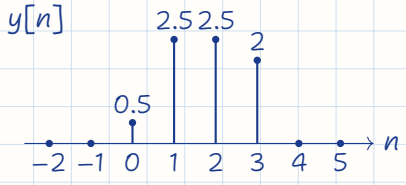
$y[0] = 0.5\,h[0] + 2\,h[-1] = 0.5$ ($h[-1] \to 0$)

$y[1] = 0.5\,h[1] + 2\,h[0] = 2.5$

$y[2] = 0.5\,h[2] + 2\,h[1] = 2.5$

$y[3] = 0.5\,h[3] + 2\,h[2] = 2$ ($h[3] \to 0$)

$y[n] = 0, \quad n \ge 4$



Örnek (Ex)

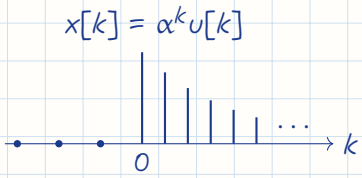
orj. s. 3

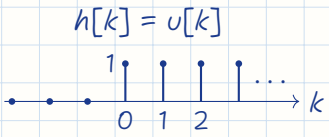
$x[n] = \alpha^n u[n], \quad h[n] = u[n], \quad y[n] = ?$ $\qquad 0 < \alpha < 1$

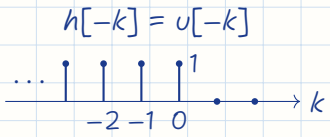
Çözüm:

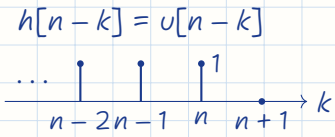
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]\,u[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}\,u[n]$$

($k \ge 0$; $n-k \ge 0 \Rightarrow n \ge k$)





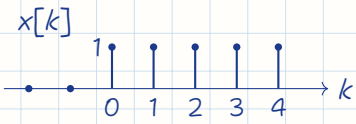


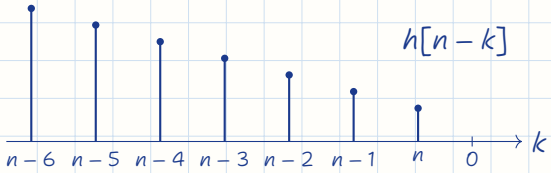


Örnek (Ex)

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{dışında} \end{cases} \qquad h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{dışında} \end{cases} \qquad y[n] = ? \quad \alpha > 1$$

Çözüm:





$n < 0$ $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = 0$

$0 \le n \le 4$ $\quad y[n] = \sum_{k=0}^{n} x[k]\,h[n-k] = \sum_{k=0}^{n} \alpha^{n-k} \underset{\uparrow\, n-k=r}{=} \sum_{r=0}^{n} \alpha^{r} = \frac{1-\alpha^{n+1}}{1-\alpha}$

$\left.\begin{matrix} n > 4 \\ n-6 \le 0 \end{matrix}\right\}$ $4 < n \le 6$ $\quad y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \alpha^{n} \sum_{k=0}^{4} \alpha^{-k} = \alpha^{n} \cdot \frac{1-(\alpha^{-1})^{5}}{1-(\alpha^{-1})} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$

$\left.\begin{matrix} n > 6 \\ n-6 \le 4 \end{matrix}\right\}$ $6 < n \le 10$ $\quad y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} \underset{\uparrow\, r=k-n+6}{=} \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^{6} \sum_{r=0}^{10-n} \alpha^{-r}$

$= \alpha^{6} \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$

$n-6 > 4 \rightarrow n > 10$ $\quad y[n] = 0$

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & n > 10 \end{cases}$$

Sürekli Zamanlı LTI Sistemlerde Konvolüsyon

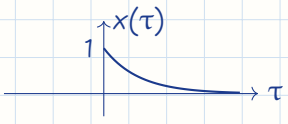
orj. s. 4

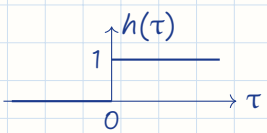
$$x(t) \longrightarrow \boxed{\overset{\text{LTI}}{h(t)}} \longrightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

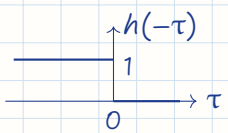
Örnek (Ex)

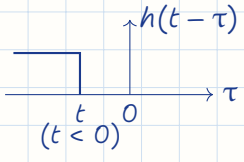
$x(t) = e^{-at}u(t),\ a > 0, \quad h(t) = u(t), \quad y(t) = ?$

Çözüm:







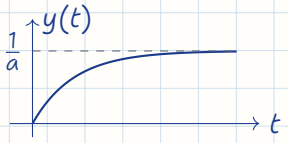


$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$t < 0 \quad y(t) = 0$

$t \geq 0 \quad y(t) = \int_0^t e^{-a\tau}\, d\tau = \left.\frac{e^{-a\tau}}{-a}\right|_0^t = \frac{1}{a}\left(1 - e^{-at}\right)$

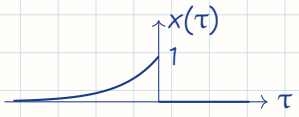
$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right) u(t)$$

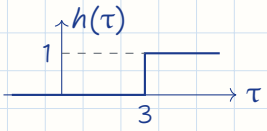


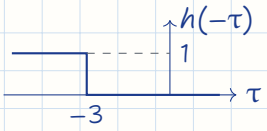
Örnek (Ex)

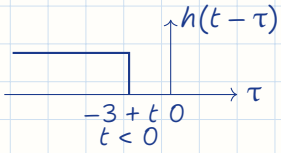
$x(t) = e^{2t}u(-t), \quad h(t) = u(t-3), \quad y(t) = ?$

Çözüm:









orj. s. 5

$t - 3 \leq 0 \quad y(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2} e^{2\tau}\Big|_{-\infty}^{t-3} = \frac{1}{2} e^{2(t-3)}$

$t - 3 > 0 \quad y(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2}$

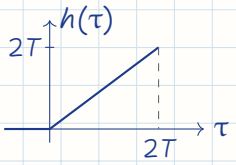
$$y(t) = \begin{cases} 0.5\, e^{2(t-3)}, & t \leq 3 \\ 0.5, & t > 3 \end{cases}$$

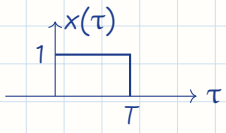
Örnek (Ex)

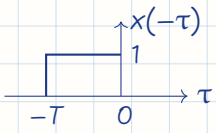
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{dışında} \end{cases}, \qquad h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{dışında} \end{cases}, \qquad y(t) = ?$$

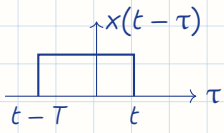
Çözüm:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$









$t < 0 \quad y(t) = 0$

$0 < t < T \quad y(t) = \int_0^t \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t = t^2/2$ ← $t > 0$, $t - T < 0$

$T < t < 2T \quad y(t) = \int_{t-T}^{T} \tau\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{T} = \frac{T^2}{2} - \frac{(t-T)^2}{2} = tT - T^2/2$ [1] ← $t < 2T$, $t - T > 0$

$2T < t < 3T \quad y(t) = \int_{t-T}^{2T} \tau\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T} = -\frac{t^2}{2} + Tt + \frac{3T^2}{2}$ ← $t > 2T$, $t - T < 2T$

$t > 3T \quad y(t) = 0$

$$y(t) = \begin{cases} 0, & t < 0 \\ t^2/2, & 0 < t < T \\ tT - T^2/2, & T < t < 2T \\ -t^2/2 + Tt + 3T^2/2, & 2T < t < 3T \\ 0, & t > 3T \end{cases}$$

Konvolüsyon işleminin özellikleri

orj. s. 6

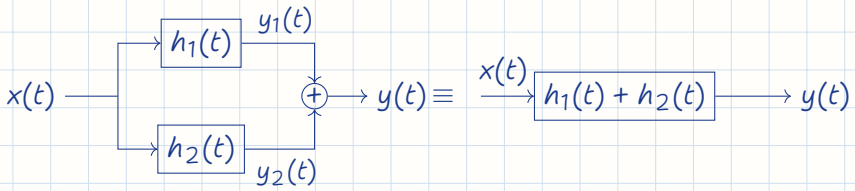
1) Yer değiştirme (commutative) özelliği

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

2) Dağılma (distributive) özelliği

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$
$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$



Dağılma özelliği ≡ Paralel Bağlama (Only for LTI)

[1] Claude notu: $T < t < 2T$ satırında muhtemelen üst sınır t olmalı: $\int_{t-T}^{t} \tau\, d\tau = \frac{t^2}{2} - \frac{(t-T)^2}{2} = tT - T^2/2$ (yazılan sonuç bununla uyumlu).

3) Birleşme (associative) özelliği

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n] = x[n] * h_1[n] * h_2[n]$$
$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t) = x(t) * h_1(t) * h_2(t)$$

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t) \equiv x(t) \longrightarrow \boxed{\begin{matrix} h_1(t) * h_2(t) \\ \text{or} \\ h_2(t) * h_1(t) \end{matrix}} \longrightarrow y(t)$$

Birleşme özelliği ≡ Seri Bağlama (Only for LTI)

LTI sistemlerin özellikleri

LTI sistemlerin özellikleri birim dürtü cevabı üzerinden belirlenebilir.

1) Hafızaya sahip olan ve olmayan LTI sistemler

(LTI systems with and without memory)

Bellekli ve belleksiz LTI sistemler

Reminder: Bir sistemin herhangi bir zamandaki çıkışı, sadece o andaki girişlere bağlı ise belleksizdir.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Belleksiz: $h[n] = 0,\ n \neq 0$

$$h[n] = K\delta[n] \longrightarrow y[n] = x[n] * h[n] = Kx[n] * \delta[n] = Kx[n]$$

Reminder: $x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[n]$

h[n]=1, n=0,1
=d[n]+d[n-1]

x[n]
y[n]=x[n]+x[n-1]

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$h(t) = 0,\ t \neq 0$$

$$h(t) = K\delta(t) \longrightarrow y(t) = Kx(t)$$

Reminder: $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$

2) Tersinir ve tersinir olmayan LTI sistemler:

orj. s. 7

Reminder: Bir sistem, tersi ile seri bağlandığında elde edilen çıktı ile girdi aynı ise sistem tersinirdir.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \boxed{h^{-1}(t)} \longrightarrow x(t) \equiv x(t) \longrightarrow \boxed{h(t) * h^{-1}(t)} \longrightarrow x(t)$$

$$h(t) * h^{-1}(t) = \delta(t)$$

Tersinir: $h(t) * h^{-1}(t) = \delta(t)$

$\Rightarrow$ LTI bir sistemin tersi de LTI'dır.

$$h[n] * h^{-1}[n] = \delta[n]$$

Örnek (Ex)

$y(t) = x(t - t_0)$ sistemi tersinir midir?

Çözüm:

$$y(t) = x(t)*h(t) = x(t-t_0) \longleftarrow h(t) = \delta(t-t_0) \longrightarrow y(t) = x(t)*\delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-t_0-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\,\delta(t-t_0-\tau)\,d\tau$$

$$= x(t-t_0)\underbrace{\int_{-\infty}^{\infty} \delta(t-t_0-\tau)\,d\tau}_{1} = x(t-t_0)$$

$$h(t) * h^{-1}(t) = \delta(t)$$

$$h^{-1}(t) = \delta(t + t_0)$$

$$\delta(t - t_0) * \delta(t + t_0) = \delta(t) \qquad \text{Tersinirdir.}$$

Örnek (Ex)

Birim dürtü cevabı $h[n] = u[n]$ olan LTI bir sistem tersinir midir?

Çözüm:

$$x[n] \rightarrow \boxed{h[n] = u[n]} \rightarrow y[n] \rightarrow \boxed{h^{-1}[n] = ?} \rightarrow w[n] = x[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

$$w[n] = y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k]$$

$$= (\cdots + x[n-1] + x[n]) - (\cdots + x[n-2] + x[n-1]) = x[n]$$

$\hookrightarrow\ h^{-1}[n] = \delta[n] - \delta[n-1]$

$$h[n] * h^{-1}[n] \stackrel{?}{=} \delta[n]$$

$$u[n] * (\delta[n] - \delta[n-1]) = u[n] * \delta[n] - u[n] * \delta[n-1] = u[n] - u[n-1] = \delta[n] \checkmark \quad \text{Tersinir.}$$