İşaretler ve Sistemler

Ders 5 – Sürekli zamanlı Fourier serileri

Orijinal not: Isaret-ve-Sistemler-ders-5.pdf

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Sürekli Zamanlı Fourier Serileri

orj. s. 1

(Continuous-Time Fourier Series)

$$x(t) = x(t + T), \ \forall t$$

Periyodik bir işaret Fourier serisine açılabilir.

Fourier Serisinin mevcut olma koşulları (Dirichlet condition)

1) $x(t)$ işareti, bir periyot boyunca mutlak integrallenebilir olmalıdır.

$$\int_0^{T_0} |x(t)|\,dt < \infty$$

Zayıf Dirichlet koşulu (weak Dirichlet condition)
Bu koşul sağlanırsa Fourier serisi vardır, ancak seri her noktada yakınsamayabilir.

2) $x(t)$ işareti, bir periyot boyunca
   - sonlu sayıda maksimum ve minimum değere sahip olmalıdır.
   - sonlu sayıda süreksizliğe sahip olmalıdır ve bu süreksizlik noktalarında $x(t)$ işareti sonlu değer almalıdır.

Strong Dirichlet condition (1 ve 2)

Bu koşullar sağlanırsa Fourier serisi vardır.

(Bu şartların sağlanmadığı durumlarda da Fourier seri açılımı olabilir.)

- Genelleştirilmiş Fourier Serisi (Generalized FS)
- Trigonometrik Fourier Serisi (Trigonometric FS)
- Compact FS
- Üstel Fourier Serisi (Exponential FS)

Ortogonal/ortonormal işaretler

orj. s. 2

$[t_1, t_2]$, $\psi_i(t)$, $-\infty < i < \infty$

$$\int_{t_1}^{t_2} \psi_n(t)\,\psi_m^*(t)\,dt = \begin{cases} E_m, & m = n \\ 0, & m \neq n \end{cases}$$

ise $\psi_n(t)$ ve $\psi_m(t)$ işaretleri birbirine diktir (ortogonal).

$E_m = 1$ ise $\psi_m(t)$ ve $\psi_n(t)$ ortonormal işaretlerdir.

Genelleştirilmiş FS açılımı

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \psi_k(t), \quad [t_1, t_2] \qquad \psi_k(t) : \text{Temel fonksiyonlar (Basis functions)}$$

$$c_k = \frac{1}{E_k} \int_{t_1}^{t_2} x(t)\, \psi_k^*(t)\, dt$$

Üstel Fourier Serisi Açılımı (Exponential FS Expansion)

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{jk\omega_0 t}, \qquad t_1 \le t \le t_1 + T_0, \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$\Rightarrow e^{jk\omega_0 t}$: Temel fonksiyonlar

$$\int_0^{T_0} e^{jk\omega_0 t} . e^{-jn\omega_0 t}\, dt = \int_0^{T_0} e^{j(k-n)\omega_0 t}\, dt = \int_0^{T_0} \cos(k-n)\omega_0 t\, dt + j \int_0^{T_0} \sin(k-n)\omega_0 t\, dt = \begin{cases} T_0, & k = n \\ 0, & k \neq n \end{cases}$$

$$\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} x_k \int_{T_0} e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t}\, dt \underset{\uparrow\, n=k}{=} T_0\, x_k \longrightarrow x_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$\Rightarrow k = 0$

$$x_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt \;:\; x(t) \text{ işaretinin bir periyodundaki ortalamasıdır.}$$

Trigonometrik Fourier Serisi Açılımı

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t, \qquad t_1 \le t \le t_1 + T_0, \quad \omega_0 = \frac{2\pi}{T_0}$$

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t)\, dt$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

orj. s. 3

$\Rightarrow \cos(k\omega_0 t)$ ve $\sin(k\omega_0 t)$: Temel fonksiyonlar

$$\int_{T_0} \cos(k\omega_0 t) \sin(n\omega_0 t)\, dt = 0$$

$$\int_{T_0} \cos(k\omega_0 t) \cos(n\omega_0 t)\, dt = \begin{cases} T_0/2, & n = k \\ 0, & n \neq k \end{cases}$$

$$\int_{T_0} \sin(k\omega_0 t) \sin(n\omega_0 t)\, dt = \begin{cases} T_0/2, & n = k \\ 0, & n \neq k \end{cases}$$

$\Rightarrow x(t)$ işareti **tek** işaret ise $a_0 = a_n = 0$

Çift işaret ise $b_n = 0$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t} = x_0 + \sum_{k=1}^{\infty} x_k e^{jk\omega_0 t} + x_{-k} e^{-jk\omega_0 t}$$

$$= x_0 + \sum_{k=1}^{\infty} (x_k + x_{-k}) \cos k\omega_0 t + j(x_k - x_{-k}) \sin k\omega_0 t$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + j b_k \sin \omega_0 t \quad \longleftarrow \quad \begin{aligned} a_0 &= x_0 \\ a_k &= x_k + x_{-k} \\ b_k &= x_k - x_{-k} \end{aligned}$$

Compact Fourier Series Expansion

$$a_k \cos k\omega_0 t + b_k \sin k\omega_0 t = c_k \cos(k\omega_0 t + \theta_k) \longleftarrow c_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = -\arctan\left(\frac{b_k}{a_k}\right)$$

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \theta_k), \qquad t_1 \le t \le t_1 + T_0$$

Fourier Spektrumu

c_k : genlik spektrumu (amplitude spectrum)
θ_k : faz spektrumu (phase spectrum)
} frequency spectra (frekans spektrumları)

$\Rightarrow x_0 = c_0 = a_0$

$$x_k = \frac{1}{2} c_k e^{j\theta_k} = \frac{a_k - jb_k}{2} \qquad x_{-k} = \frac{1}{2} c_k e^{-j\theta_k} = \frac{a_k + jb_k}{2}, \quad k = 1, 2, \ldots, \infty$$

$$\Rightarrow c_k \cos(k\omega_0 t + \theta_k) = \frac{c_k}{2}\left[e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}\right]$$

$$= \frac{c_k}{2} e^{j\theta_k} e^{jk\omega_0 t} + \frac{c_k}{2} e^{-j\theta_k} \cdot e^{-jk\omega_0 t}$$

$$= x_k e^{jk\omega_0 t} + x_{-k} e^{-jk\omega_0 t} \quad \longleftarrow \quad \begin{aligned} x_k &= \frac{c_k}{2} e^{j\theta_k} \\ x_{-k} &= \frac{c_k}{2} e^{-j\theta_k} \end{aligned}$$

orj. s. 4

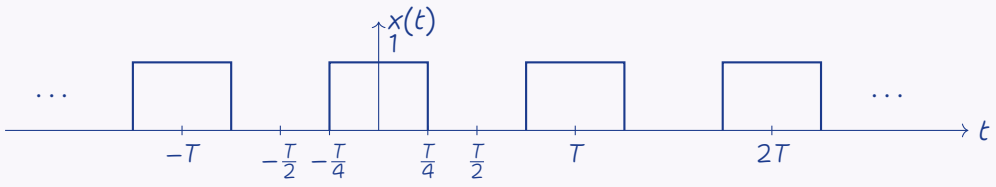
$\Rightarrow$ Üstel Fourier spektrumu

$|x_k| = |x_{-k}| = \frac{1}{2} c_k \longrightarrow$ genlik spektrumu çift simetriktir.

$\angle x_k = \theta_k, \quad \angle x_{-k} = -\theta_k \longrightarrow$ faz spektrumu tek simetriktir.

$x_k = |x_k| e^{j\theta_k} \qquad x_{-k} = |x_{-k}| e^{-j\theta_k} = |x_k| e^{-j\theta_k}$

Örnek (Ex)



Çözüm: a) Trigonometrik FS

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t, \qquad \omega_0 = \frac{2\pi}{T} = 2\pi f$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T}\int_{-T/4}^{T/4} dt = \frac{1}{T}\left[\frac{T}{4} - \left(-\frac{T}{4}\right)\right] = \frac{1}{2}$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\,dt = \frac{2}{T}\int_{-T/4}^{T/4} \cos k\omega_0 t = \frac{2}{T}\left.\frac{\sin k\omega_0 t}{k\omega_0}\right|_{-T/4}^{T/4} \underset{\uparrow\ \omega_0 T = 2\pi}{=} \frac{1}{\pi k}\left[\sin k\omega_0 \frac{T}{4} - \sin k\omega_0 \frac{-T}{4}\right]$$

$$a_k = \frac{2}{\pi k}\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0, & k \text{ çift} \\ \dfrac{2}{\pi k}, & k = 1, 5, 9, \dots \\ \dfrac{-2}{\pi k}, & k = 3, 7, 11, \dots \end{cases}$$

$b_k = 0 \longleftarrow x(t) = x(-t)$ çift işaret olduğundan

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right]$$

b) Compact FS

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \theta_k)$$

$$c_0 = a_0 = \frac{1}{2}$$

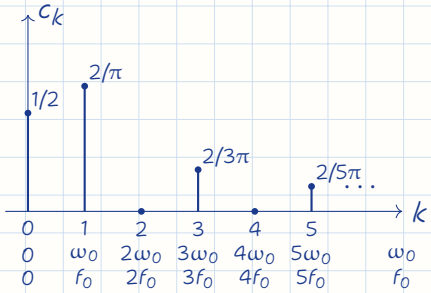
$$c_k = \sqrt{a_k^2 + b_k^2} = a_k \quad [1]$$

$$-\cos x = \cos(x - \pi)$$

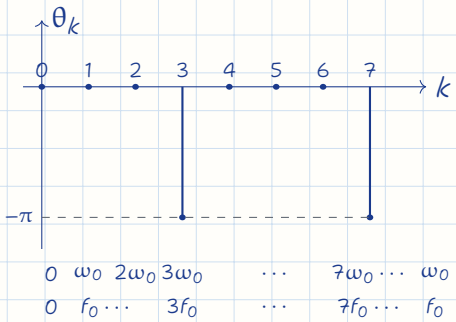
$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\omega_0 t + \frac{1}{3}\cos(3\omega_0 t - \pi) + \frac{1}{5}\cos 5\omega_0 t + \frac{1}{7}\cos(7\omega_0 t - \pi) + \cdots\right]$$

$$c_k = \begin{cases} 0, & k \text{ çift} \\ \dfrac{2}{\pi k}, & k \text{ tek} \end{cases} \qquad \theta_k = \begin{cases} 0, & k = 1, 5, 9, \dots \\ -\pi, & k = 3, 7, 11, \dots \end{cases}$$

[1] Claude notu: $b_k = 0$ olduğundan $c_k = \sqrt{a_k^2} = |a_k|$ olmalı (aşağıda da $c_k = \frac{2}{\pi k}$, yani $|a_k|$ kullanılmış).



Genlik spektrumu



Faz spektrumu

orj. s. 5

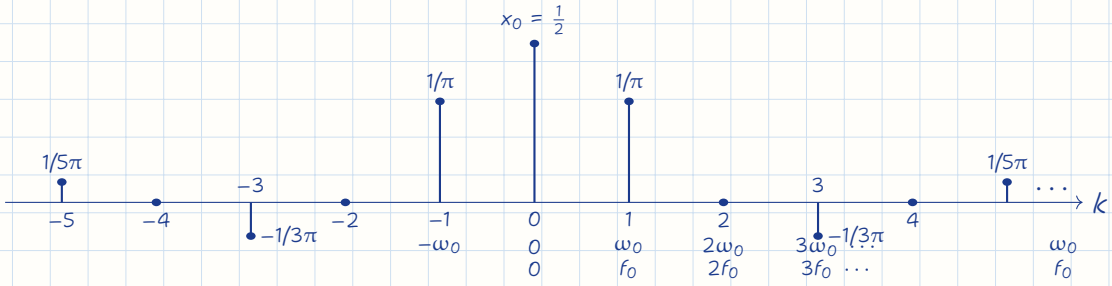
Çözüm: c) Üstel FS

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}$$

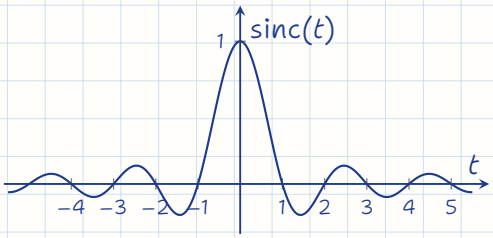
$$x_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T/4}^{T/4} e^{-jk\omega_0 t}\,dt = \frac{1}{T}\frac{e^{-jk\omega_0 t}}{-jk\omega_0} = \frac{-1}{jk\,2\pi}\left[e^{-jk\omega_0 \frac{T}{4}} - e^{-jk\omega_0 \frac{-T}{4}}\right]$$

$$x_k = \frac{1}{\pi k}\left[\frac{e^{jk\pi/2} - e^{-jk\pi/2}}{2j}\right] = \frac{1}{\pi k}\sin\left(\frac{k\pi}{2}\right)$$

$$x_0 = \frac{1}{2} \qquad x_1 = x_{-1} = \frac{1}{\pi} \qquad x_3 = x_{-3} = \frac{-1}{3\pi} \qquad x_5 = x_{-5} = \frac{1}{5\pi} \quad \cdots$$



Not: $\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

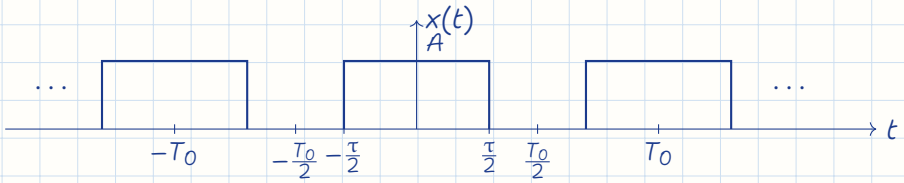


$t = k \in \mathbb{Z} - \{0\} \Rightarrow \operatorname{sinc}(t) = 0$

$$\Rightarrow x_k = \frac{1}{\pi k}\sin\left(\frac{\pi k}{2}\right) = \frac{1}{2}\operatorname{sinc}(k/2)$$

orj. s. 6

⇒ Periyodik dikdörtgen dalga işaretinin üstel Fourier spektrumunun çizimi

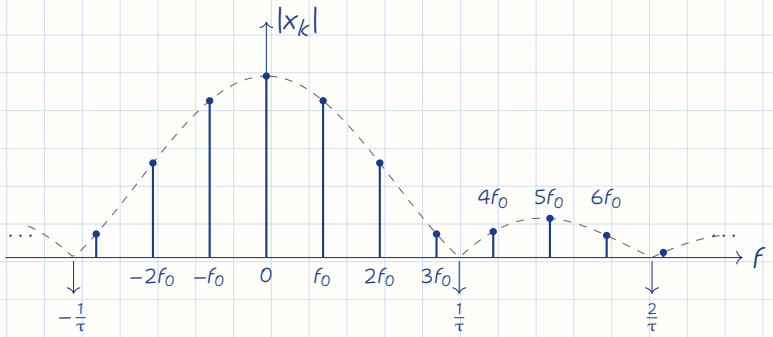


$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{jk\omega_0 t}$$

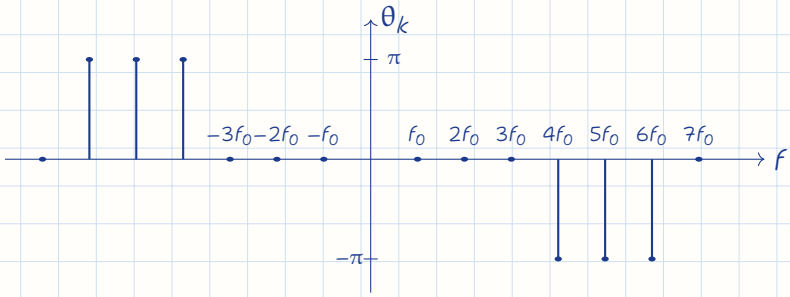
$$x_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{2A}{k\omega_0 T_0}\sin\left(k\omega_0\frac{\tau}{2}\right) \underset{\substack{\uparrow \\ \omega_0 T_0 = 2\pi \\ \omega_0 = 2\pi f_0}}{=} \frac{A}{\pi k}\sin(k\pi f_0\tau) \underset{\substack{\uparrow \\ f_0\tau \\ \text{çarp} \\ \text{böl}}}{=} Af_0\tau\,\frac{\sin(\pi k f_0\tau)}{\pi k f_0\tau}$$

$$x_k = Af_0\tau\,\mathrm{sinc}(kf_0\tau)$$

$3f_0 < 1/\tau < 4f_0$ olsun.

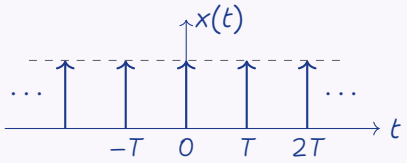


Genlik spektrumu (Çift fonksiyon)



Faz spektrumu (Tek fonksiyon)

Örnek (Ex)



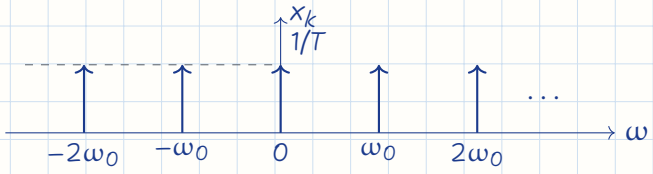
$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ dürtü katarı (impulse train)

Üstel FS açılımını bulunuz.

Çözüm:

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{jk\omega_0 t}$$

$$x_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\underbrace{\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt}_{=1} = \frac{1}{T}$$



Örnek (Ex)

orj. s. 7

$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \pi/4)$

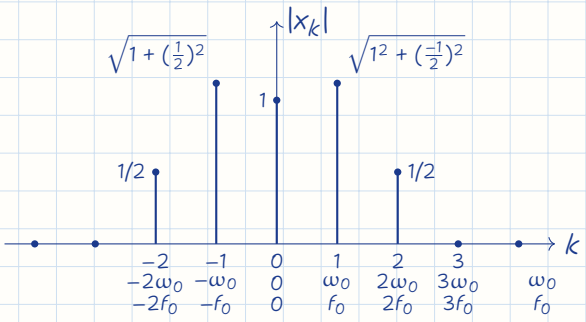
Çözüm:

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{1}{2}\left[e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right]$$

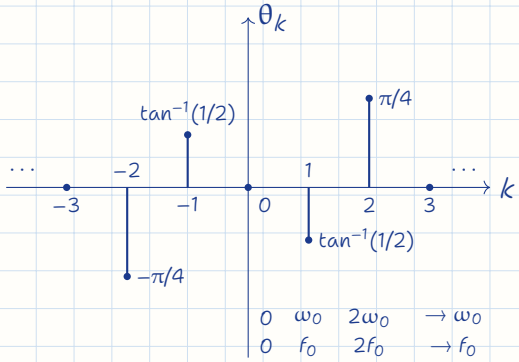
$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{j\pi/4}e^{j2\omega_0 t} + \frac{1}{2}e^{-j\pi/4}e^{-j2\omega_0 t}$$

$x_0 = 1$ $\quad x_1 = 1 + \frac{1}{2j} = 1 - \frac{1}{2}j$ $\quad x_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j)$ $\quad x_k = 0,\ |k| > 2$

$x_{-1} = 1 - \frac{1}{2j} = 1 + \frac{1}{2}j$ $\quad x_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j)$



Genlik spektrumu



Faz spektrumu