İşaretler ve Sistemler

Ders 6 – Sürekli zamanlı Fourier serisinin özellikleri

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Sürekli zamanlı Fourier serisinin özellikleri

orj. s. 1

$$x(t) = x(t + T_0), \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x(t) \overset{FS}{\longleftrightarrow} x_k$$

1) Scalar multiplication:

$$\alpha x(t) \overset{FS}{\longleftrightarrow} \alpha x_k$$

2) Lineerlik:

$$\alpha x_1(t) + \beta x_2(t) \overset{FS}{\longleftrightarrow} \alpha x_{1,k} + \beta x_{2,k}$$

$x_1(t)$ ve $x_2(t)$'nin periyodu (T_0) / frekansları (ω_0) aynı olmalıdır.

$\alpha x_1(t) + \beta x_2(t)$ işaretinin periyodu da T_0'dır.

3) Zamanda tersini alma (time reversal):

$$x(-t) \overset{FS}{\longleftrightarrow} x_{-k}$$

$\Rightarrow x(t)$ çift işaret ise FS katsayıları da çifttir.

$$x(t) = x(-t) \overset{FS}{\longleftrightarrow} x_k = x_{-k}$$

$\Rightarrow x(t)$ tek işaret ise FS katsayıları da tektir.

$$x(t) = -x(-t) \overset{FS}{\longleftrightarrow} x_k = -x_{-k}$$

4) Zamanda öteleme:

$$x(t-t_0) \overset{FS}{\longleftrightarrow} x_k\, e^{-jk\omega_0 t_0}$$

İspat: $x(t-t_0) \overset{FS}{\longleftrightarrow} d_k$ olsun

$$d_k = \frac{1}{T_0}\int_{T_0} x(t-t_0)\, e^{-jk\omega_0 t}\, dt \underset{\substack{\uparrow \\ t-t_0=\tau}}{=} \frac{1}{T_0}\int_{T_0} x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau = e^{-jk\omega_0 t_0}\underbrace{\frac{1}{T_0}\int_{T_0} x(\tau)\, e^{-jk\omega_0\tau}\, d\tau}_{=x_k}$$

$$d_k = x_k\, e^{-jk\omega_0 t_0}$$

$|d_k| = |x_k| \longrightarrow$ Periyodik bir işaretin zamanda ötelenmesi FS katsayılarının genliğini değiştirmez.

5) Frekansta öteleme:

$$x(t)\, e^{jk_0\omega_0 t} \overset{FS}{\longleftrightarrow} x_{k-k_0}$$

İspat: $d_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jk_0\omega_0 t}\cdot e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j(k-k_0)\omega_0 t}\, dt = x_{k-k_0}$

6) Zamanda ölçeklendirme:

$$x(\alpha t) \overset{FS}{\longleftrightarrow} x_k$$

Periyodik bir işaretin zamanda ölçeklendirilmesi FS katsayılarını değiştirmez. Ancak, $x(\alpha t)$ işaretinin frekansı $\alpha\omega_0$ ve periyodu $\frac{T_0}{\alpha}$ olarak değişir.

7) Çarpma:

(Frekansta konvolüsyon)

$$x_1(t)\cdot x_2(t) \overset{FS}{\longleftrightarrow} x_{1,k} * x_{2,k} = \sum_{n=-\infty}^{\infty} x_{1,n}\, x_{2,k-n}$$

8) Zamanda türev alma:

$$\frac{d^n x(t)}{dt^n} \overset{FS}{\longleftrightarrow} (jk\omega_0)^n\, x_k$$

9) Eşlenik alma (conjugation):

$$x^*(t) \overset{FS}{\longleftrightarrow} x^*_{-k}$$

İspat: $x^*(t) = \sum_{k=-\infty}^{\infty} x_k^*\, e^{-jk\omega_0 t} \underset{\substack{\uparrow \\ k=-n}}{=} \sum_{n=-\infty}^{\infty} x^*_{-n}\, e^{jn\omega_0 t}$

orj. s. 2

$\Rightarrow x(t)$ işareti reel ise FS katsayıları eşlenik simetriktir.

$$x(t) = x^*(t) \overset{FS}{\longleftrightarrow} x_k = x^*_{-k} \longrightarrow x_0 \text{ reeldir.}$$

$$|x_k| = |x_{-k}|$$

$$\theta_k = -\theta_{-k} \quad (\angle x_k = -\angle x_{-k})$$

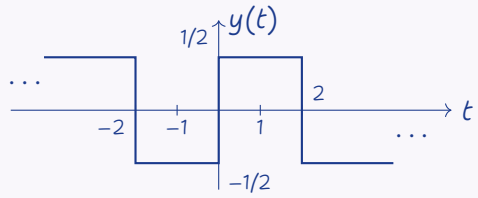
$\Rightarrow x(t)$ işareti reel ve çift ise FS katsayıları da reel ve çifttir.

$$x(t) = x^*(t) = x(-t) \overset{FS}{\longleftrightarrow} x_k = x_{-k}^* = x_{-k} \longrightarrow x_k^* = x_k$$

$\Rightarrow x(t)$ işareti reel ve tek ise FS katsayıları sanal ve tektir.

$$x(t) = x^*(t) = -x(-t) \overset{FS}{\longleftrightarrow} x_k = x_{-k}^* = -x_{-k} \longrightarrow x_k^* = -x_k$$
$$\hookrightarrow x_0 = 0$$

Örnek (Ex)



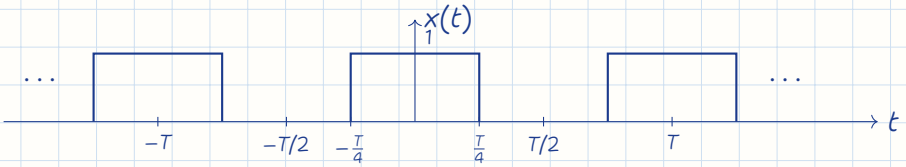
$y(t)$ işaretinin Fourier Serisi katsayılarını bulunuz.

Çözüm: I. yol:

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t}, \qquad T_0 = 4, \quad \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$$

$$y_k = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-jk\omega_0 t}\, dt$$

II. yol:



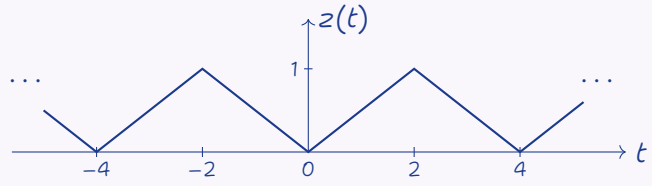
$$x_0 = \frac{1}{2} \qquad x_k = \frac{\sin(k\pi/2)}{\pi k}, \quad k \neq 0$$

$$T = 4 \Rightarrow y(t) = x(t-1) - 1/2$$

$$y_k = \begin{cases} x_0 - 1/2, & k = 0 \quad \longleftarrow -1/2 \text{ dc offset} \\ x_k\, e^{-jk\omega_0 t_0} = x_k\, e^{-jk\pi/2}, & k \neq 0 \quad \longleftarrow x(t-1) \text{ zamanda kaydırma.} \end{cases}$$

$$y_k = \begin{cases} 0, & k = 0 \\ \dfrac{\sin(k\pi/2)\, e^{-jk\pi/2}}{k\pi}, & k \neq 0 \end{cases}$$

Örnek (Ex)



Çözüm:

$$z(t) = |t|/2, \quad |t| < 2$$

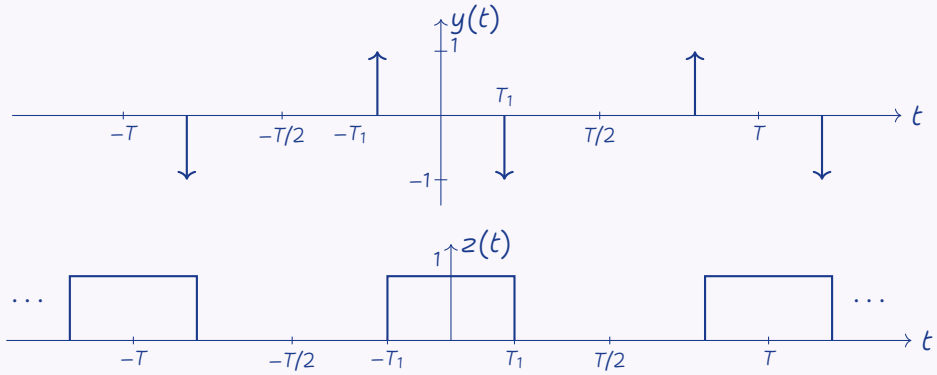
$$\frac{dz(t)}{dt} = y(t) \overset{FS}{\longleftrightarrow} y_k = jk\omega_0 z_k \Rightarrow z_k = \frac{1}{jk\omega_0} y_k = \underset{\uparrow \\ \omega_0 = \pi/2}{=} \frac{2}{jk\pi} y_k$$

$$z_k = \frac{2}{j(k\pi)^2} \sin(k\pi/2)\, e^{-jk\pi/2}, \quad k \neq 0$$

$$z_0 = \frac{1}{T_0} \int_{T_0} z(t)\, dt = \frac{1}{2}$$

Örnek (Ex)

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ dürtü treni olup $x_k = \frac{1}{T}$'dir. Aşağıda verilen işaretlerin FS katsayılarını bulunuz.



Çözüm:

$$y(t) = x(t + T_1) - x(t - T_1) \overset{FS}{\longleftrightarrow} y_k = x_k e^{jk\omega_0 T_1} - x_k e^{-jk\omega_0 T_1} = \frac{1}{T}\left[e^{+jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right]$$

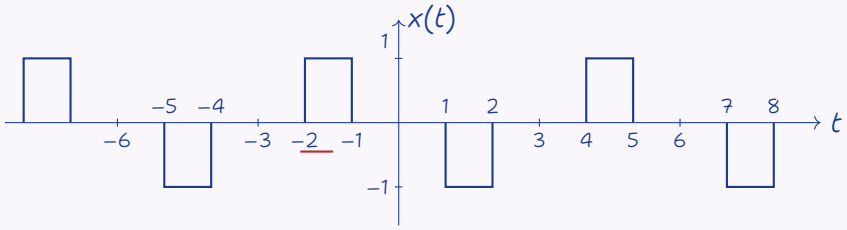
$$y_k = \frac{2j}{T} \sin(k\omega_0 T_1)$$

$$\frac{dz(t)}{dt} = y(t) \overset{FS}{\longleftrightarrow} y_k = jk\omega_0 z_k \longrightarrow z_k = \frac{1}{jk\omega_0} y_k = \frac{2j}{Tjk\omega_0} \sin(k\omega_0 T_1)$$

$$z_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

$$z_0 = \frac{1}{T_0} \int_{T_0} z(t)\, dt = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

Örnek (Ex)



Çözüm:

$$x(t) = z(t + 3/2) - z(t - 3/2) \quad \text{ve} \quad T_1 = 1/2$$

$$x_k = z_k e^{+jk\omega_0 3/2} - z_k e^{-jk\omega_0 3/2}, \qquad T = 6 \longrightarrow \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$$

$$x_k = z_k \left[e^{jk\pi/2} - e^{-jk\pi/2} \right] = 2j\, z_k \sin(k\pi/2)$$

$$x_k = 2j \frac{1}{k\pi} \sin(k\pi/6)\, \sin(k\pi/2)$$

$$x_k = \frac{\cos(2k\pi/3) - \cos(k\pi/3)}{jk\pi}$$

Sürekli Zamanlı LTI sistemlerin Periyodik İşaretlere Cevabı

$$Ae^{st} \longrightarrow \boxed{h(t)} \longrightarrow Ae^{st} \cdot H(s), \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

orj. s. 4

⇒ Bir sistemin çıktısı, girdinin bir sabit ile çarpımı ise, o girdi işaretine sistemin özfonksiyonu (eigenfunction), ilgili sabite (genlik çarpanına) özdeğeri (eigenvalue) denir.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, Ae^{s(t-\tau)}\, d\tau = Ae^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{\triangleq H(s)} = Ae^{st} H(s) \longrightarrow Ae^{st} : \text{özfonksiyon}$$

$$\hookrightarrow H(s) : \text{özdeğer}$$

⇒ $s = j\omega$

$$Ae^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow Ae^{j\omega t} H(j\omega)$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \ : \text{ Frequency Response (Frekans cevabı)}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} \underbrace{x_k H(jk\omega_0)}_{\triangleq y_k} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} y_k\, e^{jk\omega_0 t}$$

Bir LTI sistemin girişinde hangi frekans bileşenleri varsa çıkışında da o frekans bileşenleri vardır.

Örnek (Ex)

$$x(t) = \sum_{k=-3}^{3} x_k\, e^{jk2\pi t}, \quad x_0 = 1,\ x_1 = x_{-1} = \frac{1}{4},\ x_2 = x_{-2} = \frac{1}{2},\ x_3 = x_{-3} = \frac{1}{3}$$

a) $x(t) = ?$

b) $h(t) = e^{-t}u(t)$ ise $y(t) = ?$

Çözüm: a)

$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t} \right] + \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t} \right] + \frac{1}{3}\left[e^{j6\pi t} + e^{-j6\pi t} \right]$$

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

b)

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} x_k H(jk\omega_0) e^{jk\omega_0 t}, \qquad \omega_0 = 2\pi$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-t} \cdot e^{-j\omega t} dt = \int_0^{\infty} e^{-(1+j\omega)t} dt = \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Big|_0^{\infty} = \frac{1}{1+j\omega}$$

$$y_k = x_k H(jk\omega_0) = x_k H(jk2\pi)$$
$$y_0 = x_0 = 1$$
$$y_1 = x_1 H(j2\pi) = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right)$$
$$y_{-1} = x_{-1} H(-j2\pi) = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right)$$

$$y_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right) \qquad y_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right) \qquad y_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right) \qquad y_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

Örnek (Ex)

orj. s. 5

$x(t) = \cos(2t + \pi/4), \quad x_k = ? \quad P_x = ?$

Çözüm:

$$x(t) = \frac{1}{2}\left[e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}\right] = \frac{1}{2}e^{j\pi/4} \cdot e^{j2t} + \frac{1}{2}e^{-j\pi/4} \cdot e^{-j2t}, \qquad \omega_0 = 2 \rightarrow T = \frac{2\pi}{\omega_0} = \pi$$

$$x_1 = \frac{1}{2}e^{j\pi/4} \qquad x_{-1} = \frac{1}{2}e^{-j\pi/4}$$

$$P_x = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |x_k|^2 = \left(\frac{1}{2}\right)^2 \cdot 2 = \frac{1}{2}$$

Örnek (Ex)

$x_k = \begin{cases} jk, & |k| \le 3 \\ 0, & \text{dışında} \end{cases}$ ise $x(t) = ?$

Çözüm:

$$x(t) = \sum_{k=-3}^{3} x_k e^{jk\omega_0 t} \longleftarrow \begin{aligned} x_0 &= 0 \\ x_1 &= j \quad x_{-1} = -j \\ x_2 &= 2j \quad x_{-2} = -2j \\ x_3 &= 3j \quad x_{-3} = -3j \end{aligned}$$

$$x(t) = j\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + 2j\left[e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right] + 3j\left[e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right]$$
$$x(t) = -2\sin\omega_0 t - 4\sin(2\omega_0 t) - 8\sin(3\omega_0 t)$$ [1]

[1] Claude notu: muhtemelen $-6\sin(3\omega_0 t)$ olmalı ($3j \cdot 2j \sin 3\omega_0 t = -6\sin 3\omega_0 t$).

Örnek (Ex)

$$H(j\omega) = \begin{cases} 1, & |\omega| \le 100 \\ 0, & |\omega| > 100 \end{cases}$$

$x(t)$'nin periyodu $T = \pi/6$
$y(t) = x(t)$ olması için hangi k değerleri için $x_k = 0$ olmalıdır.

Çözüm:

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} x_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$\left.\begin{aligned} &|k\omega_0| \le 100 \\ &\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi/6} = 12 \end{aligned}\right\} \; |k| \le \frac{100}{12} = 8.3 \;\rightarrow\; |k| \le 8 \;\rightarrow\; |k| > 8$$

Örnek (Ex) ★

Bir $x(t)$ işareti, aşağıda listelenen özellikleri ile ($\mp$ işareti dışında) belirlenebilir mi?

1. Reeldir.
2. $T_0 = 4$ periyodu ile periyodiktir ve FS katsayıları x_k'dır.
3. $x_k = 0$, $|k| > 1$.
4. FS katsayıları $y_k = e^{-jk\pi/2} \cdot x_{-k}$ olan $y(t)$ sinyali tektir.
5. $\frac{1}{4}\int_4 |x(t)|^2 \, dt = 1/2$.

Çözüm: **2.** $T_0 = 4 \longrightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$

3. $x_k = 0,\ |k| > 1 \longrightarrow x(t) = \sum_{k=-1}^{1} x_k e^{jk\omega_0 t} = x_0 + x_1 e^{jk\pi t/2} + x_{-1} e^{-jk\pi t/2}$

1. $x(t)$ Reel $\longrightarrow x(t) = x^*(t) \Rightarrow x_k = x^*_{-k}$ ve x_0 Reel

$$x(t) = x_0 + x_1 e^{jk\omega_0 t} + \left(x_1 e^{jk\omega_0 t}\right)^* = x_0 + 2\,\mathrm{Re}\left\{x_1 e^{jk\pi t/2}\right\}$$

4. $y_k = e^{-jk\pi/2} \cdot x_{-k} \longrightarrow x(t) \overset{FS}{\longleftrightarrow} x_k$ ise $x(-t) \overset{FS}{\longleftrightarrow} x_{-k}$

$$x(t - t_0) \overset{FS}{\longleftrightarrow} x_k e^{-jk\omega_0 t_0}$$

$y(t) = x(-t+1)$ reel ve tektir. $\longrightarrow y_k$ sanal ve tek $\longrightarrow y_0 = 0$

$\hookrightarrow y_k = -y_{-k}$ $\quad (y_1 = -y_{-1})$

orj. s. 6

5.

$$\frac{1}{4}\int_4 |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |x_k|^2 = \sum_k |y_k|^2 = |y_0|^2 + |y_1|^2 + |y_{-1}|^2$$

$$2|y_1|^2 = \frac{1}{2} \longrightarrow |y_1| = \frac{1}{2}$$

y_k sanal olacağından $y_1 = \mp j/2$

$$x_{-k} = y_k e^{jk\pi/2} \begin{cases} \longrightarrow x_0 = 0 \\ \longrightarrow y_1 = j/2 \Rightarrow \left.\begin{aligned} x_{-1} &= \frac{j}{2} e^{j\pi/2} = \frac{-1}{2} \\ x^*_{-1} &= x_1 = \frac{-1}{2} \end{aligned}\right\} x(t) = 2\,\mathrm{Re}\left\{\tfrac{-1}{2} e^{j\pi t/2}\right\} = -\cos(\pi t/2) \\ \longrightarrow y_1 = -j/2 \Rightarrow x_{-1} = \frac{1}{2} = x_1 \longrightarrow x(t) = \cos(\pi t/2) \end{cases}$$