İşaretler ve Sistemler

Ders 7 – Ayrık zamanlı Fourier serileri

Orijinal not: Isaret-ve-Sistemler-ders-7.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Ayrık Zamanlı Fourier Serileri

orj. s. 1

(Discrete-Time Fourier Series)

$$x[n] = x[n+N],\ \forall n, \qquad N = \frac{2\pi}{\omega_0}, \qquad \omega_0 = \frac{2\pi}{N}$$

Periyodik bir işaret Fourier serisine açılabilir.

$$x[n] = \sum_{k=\langle N\rangle} x_k\, e^{jk\omega_0 n}$$

$$x_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\omega_0 n}$$

$\Rightarrow$ Ayrık zamanlı periyodik bir işaretin Fourier seri açılımı sonlu seri olup, yakınsaklık problemi yoktur.

Reminder: Ayrık zamanlı karmaşık üstel işaretler ($e^{j\omega n}$) 2π'nin katları olan frekanslarda aynıdır.

Temel fonksiyon: $e^{jk\omega_0 n} = \psi_k[n],\ k = 0, \mp1, \mp2, \ldots$

$\psi_k[n]$, $\omega_0 = 2\pi/N$ temel frekansının $k\omega_0$ şeklinde katları olan frekanslara sahip olduğundan harmonik olarak ilişkilidir ve burada N farklı işaret vardır.

$\psi_k[n] = \psi_{k+N}[n]$

İspat:

$$e^{jk\omega_0 n} \stackrel{?}{=} e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} \cdot e^{jN\omega_0 n} \underset{\uparrow\ N\omega_0 = 2\pi}{=} e^{jk\omega_0 n} \cdot \underbrace{e^{j2\pi n}}_{1} = e^{jk\omega_0 n}$$

$$\Rightarrow \sum_{k=\langle N\rangle} e^{jk\omega_0 n} = \begin{cases} N, & k = 0, \mp N, \mp 2N, \ldots \\ 0, & \text{dışında} \end{cases}$$ [1]

Karmaşık üstel işaretler sabit olmadıkça bir periyottaki değerlerinin toplamı sıfırdır.

[1] Claude notu: toplam indisi muhtemelen $n = \langle N\rangle$ olmalı (koşul k üzerinde; aşağıdaki ispat da n üzerinden toplar).

İspat:

$$\sum_{n=q}^{q+N-1} e^{jk\omega_0 n} = \sum_{n=0}^{N-1} e^{jk\omega_0(q+n)} = e^{jk\omega_0 q}\sum_{n=0}^{N-1} e^{jk\omega_0 n}$$

$$= e^{jk\omega_0 q}\left[\frac{1-e^{jk\omega_0 N}}{1-e^{jk\omega_0}}\right] \underset{\omega_0 N = 2\pi}{\overset{\uparrow}{=}} 0, \qquad k \neq pN$$

$$k = pN \longrightarrow \sum_{n=q}^{q+N-1} e^{jpN\omega_0 n} = \sum_{n=q}^{q+N-1} e^{j2\pi pn} = \sum_{n=q}^{q+N-1} 1 = N$$

$$\Rightarrow x[n] = \sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n} \qquad \longleftarrow \quad e^{-jr\omega_0 n} \text{ çarpıp, } N \text{ terimli toplam}$$

$$\sum_{n=\langle N\rangle} x[n]e^{-jr\omega_0 n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n}\cdot e^{-jr\omega_0 n} = \sum_{k=\langle N\rangle} x_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-r)\omega_0 n}}_{\begin{cases} N, & k=r \\ 0, & \text{dışında}\end{cases}} = x_r \cdot N$$

$$x_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\,e^{-jr\omega_0 n}$$

orj. s. 2

$\Rightarrow$ Ayrık zamanlı Fourier serisi açılımında, FS katsayıları x_k'larda N periyodu ile periyodiktir.

$$x_k = x_{k+N}$$

İspat:

$$x[n] = \sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} x_k \psi_k[n]$$

$$= x_0\psi_0[n] + x_1\psi_1[n] + \cdots + x_{N-1}\psi_{N-1}[n] \qquad \longleftarrow \quad k = 0,1,2,\ldots,N-1$$

$$= x_1\psi_1[n] + \cdots + x_{N-1}\psi_{N-1}[n] + x_N\psi_N[n] \qquad \longleftarrow \quad k = 1,2,\ldots,N$$

$\psi_0[n] = \psi_N[n]$ olduğundan $x_0 = x_N$ olmalıdır.

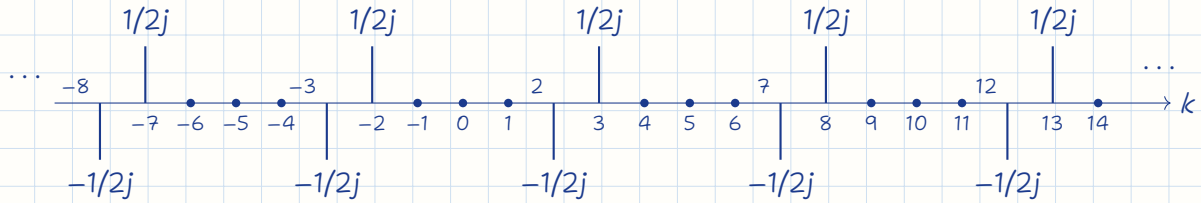
Örnek

Ex: $x[n] = \sin 3\omega_0 n$ işaretinin FS katsayılarını bulunuz ve $N = 5$ için Fourier spektrumlarını çiziniz.

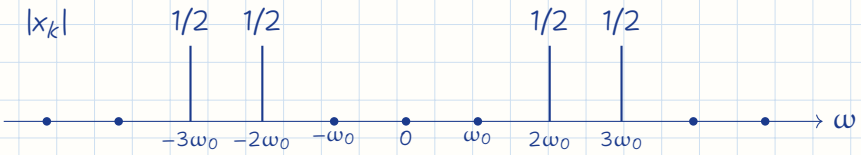
$$\left.\begin{aligned} x[n] &= \frac{1}{2j}\left[e^{j3\omega_0 n} - e^{-j3\omega_0 n}\right] \\ x[n] &= \sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n}\end{aligned}\right\} \quad k = \mp 3 \longrightarrow \begin{aligned} x_3 &= \frac{1}{2j} = -j/2 \\ x_{-3} &= -\frac{1}{2j} = j/2\end{aligned}$$

$$x_k = |x_k|e^{j\theta_k} \longrightarrow \begin{cases} x_3 = -j/2 \longrightarrow |x_3| = 1/2 \text{ ve } \theta_3 = -\pi/2 \\ x_{-3} = j/2 \longrightarrow |x_{-3}| = 1/2 \text{ ve } \theta_{-3} = \pi/2\end{cases}$$

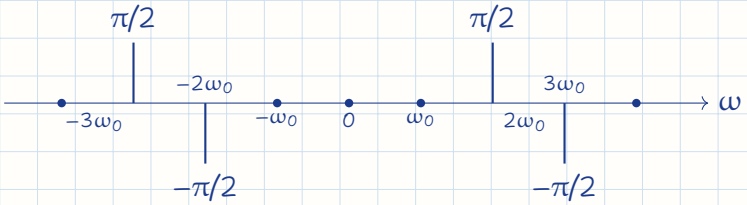
$$x_k = x_{k+rN} \xrightarrow[N=5]{} \begin{aligned} x_3 &= x_8 = x_{13} = \cdots \\ x_{-3} &= x_2 = x_7 = \cdots\end{aligned}$$



x_k, $N = 5$



Genlik spektrumu



Faz spektrumu

Örnek

orj. s. 3

Ex: $x[n] = 1 + \sin\left(\frac{2\pi}{N}n\right) + 3\cos\left(\frac{2\pi}{N}n\right) + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$

$\omega_0 = 2\pi/N$

$$x[n] = 1 + \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] + \frac{3}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j(2\omega_0 n + \pi/2)} + e^{-j(2\omega_0 n + \pi/2)}\right]$$

$x_0 = 1$

$$\left.\begin{aligned} x_1 &= \frac{1}{2j} + \frac{3}{2} = \frac{3}{2} - \frac{1}{2}j \\ x_{-1} &= -\frac{1}{2j} + \frac{3}{2} = \frac{3}{2} + \frac{1}{2}j \end{aligned}\right\} \; x_1^* = x_{-1}$$

$$\longrightarrow |x_1| = \sqrt{(3/2)^2 + (1/2)^2} = \sqrt{10}/2$$

$$\theta_1 = \tan^{-1}\left(\frac{-1/2}{3/2}\right) = -\tan^{-1}(1/3)$$

$$\longrightarrow |x_{-1}| = \sqrt{10}/2$$

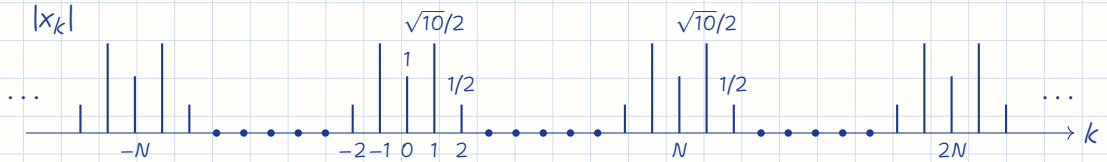
$$\theta_{-1} = \tan^{-1}(1/3)$$

$$\left.\begin{aligned} x_2 &= \frac{1}{2}e^{j\pi/2} = \frac{1}{2}j \\ x_{-2} &= \frac{1}{2}e^{-j\pi/2} = -\frac{1}{2}j \end{aligned}\right\} \; x_2^* = x_{-2}$$

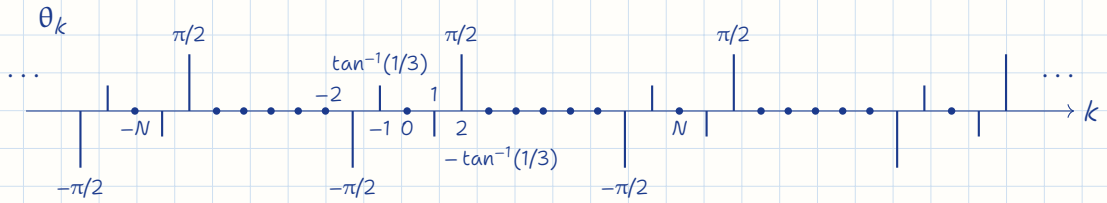
$$\longrightarrow |x_2| = |x_{-2}| = 1/2$$

$$\theta_2 = \pi/2 \qquad \theta_{-2} = -\pi/2$$

$\Rightarrow x[n]$ işareti reel olduğundan $x_k^* = x_{-k}$'dır.



Genlik spektrumu

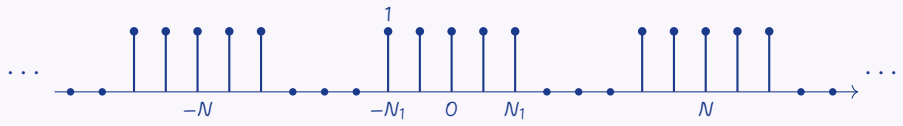


Faz spektrumu

Örnek

orj. s. 4

Ex:



Ayrık zamanlı periyodik dikdörtgen dalga işaretinin FS katsayılarını bulunuz.

$$x_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\,e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{2N_1} e^{-jk\omega_0 (n-N_1)}$$

$$x_k = \frac{1}{N} e^{jk\omega_0 N_1}\sum_{n=0}^{2N_1} e^{-jk\omega_0 n} \qquad \longleftarrow \text{ Reminder: } \sum_{i=0}^{M} a^i = \frac{1-a^{N+1}}{1-a}$$ [2]

$$x_k = \frac{1}{N} e^{jk\omega_0 N_1}\left[\frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{-jk\omega_0}}\right] \qquad \longleftarrow \text{ Reminder: } 1-e^{-j\theta} = e^{-j\theta/2}\left[e^{j\theta/2}-e^{-j\theta/2}\right]$$

$$x_k = \frac{1}{N} e^{jk\omega_0 N_1}\frac{e^{-jk\omega_0(N_1+1/2)}\left[e^{jk\omega_0(N_1+1/2)} - e^{-jk\omega_0(N_1+1/2)}\right]}{e^{-jk\omega_0/2}\left[e^{jk\omega_0/2} - e^{-jk\omega_0/2}\right]}$$

$$x_k = \frac{1}{N}\frac{\sin\left(k\omega_0(N_1+1/2)\right)}{\sin(k\omega_0/2)}, \qquad k \neq 0, \mp N, \mp 2N, \ldots$$

$$x_k = \frac{2N_1+1}{N}, \qquad k = 0, \mp N, \mp 2N, \ldots$$

Parseval Teoremi

$$P_x = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |x_k|^2$$

Ayrık zamanlı periyodik bir işaretin ortalama gücü, bütün harmonik bileşenlerin (ki bunlar N tanedir) ortalama güçlerinin toplamına eşittir.

Ayrık Zamanlı Fourier Serisinin Özellikleri

$$x[n] = x[n+N], \qquad \omega_0 = \frac{2\pi}{N}$$

$$x[n] \overset{FS}{\longleftrightarrow} x_k$$

[2] Claude notu: üst sınır M olduğuna göre muhtemelen $1-a^{M+1}$ olmalı.

1) Çarpma: $x_1[n] \cdot x_2[n] \overset{FS}{\longleftrightarrow} \sum_{\ell=\langle N\rangle} x_{1,\ell} \cdot x_{2,k-\ell}$

İspat:

$$x_1[n]x_2[n] = \sum_{k=\langle N\rangle} x_{1,k}\, e^{jk\omega_0 n} \sum_{\ell=\langle N\rangle} x_{2,\ell}\, e^{j\ell\omega_0 n}$$

$$= \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} x_{1,k}\, x_{2,\ell}\, e^{j(k+\ell)\omega_0 n} \underset{k+\ell=r}{=} \sum_{k=0}^{N-1}\sum_{r=k}^{k+N-1} x_{1,k}\, x_{2,r-k}\, e^{jr\omega_0 n}$$

$$= \sum_{r=\langle N\rangle} \underbrace{\sum_{k=\langle N\rangle} x_{1,k}\, x_{2,r-k}}_{d_r} \cdot e^{jr\omega_0 n}$$

$$= \sum_{r=\langle N\rangle} d_k\, e^{jr\omega_0 n} \quad [3]$$

orj. s. 5

2) İlk fark (first difference): $x[n] - x[n-1] \overset{FS}{\longleftrightarrow} \left(1 - e^{-jk\omega_0}\right) x_k$

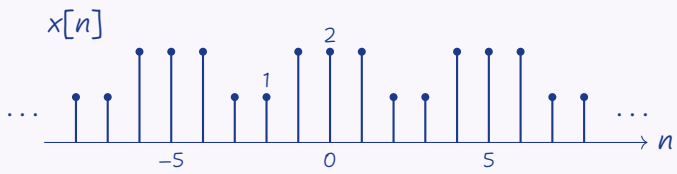
İspat:

$$x[n-1] = \sum_{k=\langle N\rangle} x_k\, e^{jk\omega_0(n-1)} = e^{-jk\omega_0} \sum_{k=\langle N\rangle} x_k\, e^{jk\omega_0 n} = e^{-jk\omega_0} x[n]$$

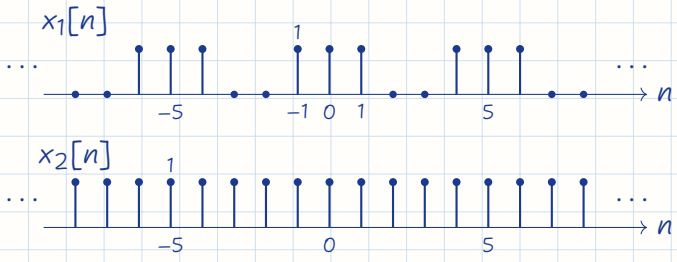
$$x[n] - x[n-1] = x[n] - e^{-jk\omega_0}x[n] = \left(1 - e^{-jk\omega_0}\right) x[n] \overset{FS}{\longleftrightarrow} \left(1 - e^{-jk\omega_0}\right) x_k$$

Örnek

Ex: FS katsayılarını bulunuz.



$N = 5$



$\Rightarrow N_1 = 1$ ve $N = 5$ olan periyodik dikdörtgen dalga ($x_1[n]$)

$x[n] = x_1[n] + x_2[n]$

$$x_{1,k} = \begin{cases} \frac{1}{N}\frac{\sin(k\omega_0(N_1+1/2))}{\sin(k\omega_0/2)}, & k \neq 0, \mp N, \mp 2N, \ldots \\ \frac{2N_1+1}{N}, & k = 0, \mp N, \mp 2N \end{cases} \underset{\substack{N=5\\N_1=1}}{=} \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & k \neq 0, \mp 5, \ldots \\ 3/5, & k = 0, \mp 5, \mp 10, \ldots \end{cases}$$

$$x_{2,k} = \frac{1}{N}\sum_{k=\langle N\rangle} x_2[n]\, e^{jk\omega_0 n} \quad [4] \xrightarrow[N=5]{k=0} x_{2,0} = \frac{1}{5}\sum_{k=0}^{4} x_2[n] = 1 \Rightarrow x_{2,k} = \begin{cases} 1, & k = 0, \mp 5, \mp 10, \ldots \\ 0, & k \neq 0, \mp 5, \mp 10, \ldots \end{cases}$$

[3] Claude notu: muhtemelen d_r olmalı.

$$x[n] = x_1[n] + x_2[n] \overset{FS}{\longleftrightarrow} x_{1,k} + x_{2,k}$$

$$x_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & k \neq 0, \mp 5, \mp 10, \dots \\ 8/5, & k = 0, \mp 5, \mp 10, \dots \end{cases}$$

Örnek

Ex: Bir $x[n]$ işareti, aşağıda verilen özellikler ile belirlenebilir mi?

1) $N = 6$ periyodu ile periyodiktir. $\boxed{\omega_0 = 2\pi/N = \pi/3}$
2) $\sum_{n=0}^{5} x[n] = 2$
3) $\sum_{n=2}^{7} (-1)^n x[n] = 1$
4) $x[n]$, yukarıdaki üç özelliği sağlayan işaretler arasında minimum güce sahiptir.

orj. s. 6

2) $x_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$ olup $x_0 = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] \longrightarrow Nx_0 = \sum_{n=\langle N\rangle} x[n]$

$Nx_0 = 2 \longrightarrow x_0 = 2/N = 1/3$

3) $Nx_k = \sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = 1$

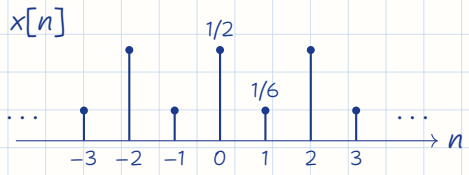
$(-1)^n = e^{-j\pi n} \longrightarrow e^{-j\pi n} = e^{-jk\omega_0 n} \longrightarrow k = \frac{\pi}{\omega_0} = \frac{\pi}{\frac{2\pi}{6}} = 3$

$x_3 = 1/6$

4) $P_x = \sum_{k=\langle N\rangle} |x_k|^2 = \sum_{k=0}^{5} |x_k|^2 \longleftarrow x_0 = 1/3$ ve $x_3 = 1/6$

Minimum güçlü olabilmesi için $x_1 = x_2 = x_4 = x_5 = 0$ seçilir.

$$x[n] = \sum_{k=\langle N\rangle} x_k e^{jk\omega_0 n} = x_0 + x_3 e^{j3\omega_0 n} \underset{\omega_0=\pi/3}{=} x_0 + x_3 e^{j\pi n} = \frac{1}{3} + \frac{1}{6}(-1)^n$$



Ayrık zamanlı LTI sistemlerin periyodik işaretlere cevabı

$$Az^n \longrightarrow \boxed{h[n]} \longrightarrow Az^n H(z), \qquad H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

[4] Claude notu: muhtemelen $\frac{1}{N}\sum_{n=\langle N\rangle} x_2[n]e^{-jk\omega_0 n}$ olmalı ($k = 0$ için sonuç değişmez).

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\,Az^{n-k} = Az^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\,z^{-k}}_{\triangleq H(z)} = Az^n H(z) \longrightarrow Az^n : \text{öz fonksiyon}$$

$\hookrightarrow H(z)$: özdeğer

$\Rightarrow z = e^{j\omega}$

$$Ae^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow Ae^{j\omega n} H(e^{j\omega})$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\omega n} \text{ : Frekans cevabı (frequency response)}$$

$$x[n] = \sum_{k=\langle N\rangle} x_k\,e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=\langle N\rangle} x_k\,H(e^{jk\omega_0})\,e^{jk\omega_0 n} \underset{\uparrow\; y_k \triangleq x_k H(e^{jk\omega_0})}{=} \sum_{k=\langle N\rangle} y_k\,e^{jk\omega_0 n}$$

Örnek

orj. s. 7

Ex: $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$ ve $x[n] = \cos\left(\frac{2\pi}{N} n\right)$ ise $y[n] = ?$

$$x[n] = \cos\left(\frac{2\pi}{N} n\right) \underset{\uparrow\; \frac{2\pi}{N} = \omega_0}{=} \cos(\omega_0 n) = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] \longrightarrow x_1 = x_{-1} = \frac{1}{2}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n$$

$\longleftarrow$ Reminder: $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, $|a| < 1$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

$$y_k = x_k H(e^{jk\omega_0}) \longrightarrow \begin{cases} y_1 = x_1 H(e^{j\omega_0}) = \dfrac{1}{2\left(1 - \alpha e^{-j\omega_0}\right)} \\ y_{-1} = x_{-1} H(e^{-j\omega_0}) = \dfrac{1}{2\left(1 - \alpha e^{j\omega_0}\right)} \end{cases}$$

$$y[n] = \sum_{k=\langle N\rangle} y_k\,e^{jk\omega_0 n} = y_1\,e^{j\omega_0 n} + y_{-1}\,e^{-j\omega_0 n}$$

$$y[n] = \frac{1}{2}\left(\frac{1}{1 - \alpha e^{-j\omega_0}}\right) e^{j\omega_0 n} + \frac{1}{2}\left(\frac{1}{1 - \alpha e^{j\omega_0}}\right) e^{-j\omega_0 n}$$

$$\frac{1}{1 - \alpha e^{-j\omega_0}} = re^{j\theta}$$

$$y[n] = \frac{1}{2} re^{j\theta} e^{j\omega_0 n} + \frac{1}{2} re^{-j\theta} e^{-j\omega_0 n} = \frac{r}{2}\left[e^{j(\omega_0 n + \theta)} + e^{-j(\omega_0 n + \theta)}\right] = r\cos(\omega_0 n + \theta)$$

Ayrık zamanlı periyodik dikdörtgen dalga işaretinin FS katsayılarını bulunuz.

$$x_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\,e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=0}^{2N_1} e^{-jk\omega_0 (n - N_1)}$$

$$x_k = \frac{1}{N} e^{jk\omega_0 N_1} \sum_{n=0}^{2N_1} e^{-jk\omega_0 n} \qquad \longleftarrow \text{Reminder: } \sum_{i=0}^{M} a^i = \frac{1-a^{N+1}}{1-a}$$

$$x_k = \frac{1}{N} e^{jk\omega_0 N_1} \left[\frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{-jk\omega_0}} \right] \qquad \longleftarrow \text{Reminder: } 1-e^{-j\theta} = e^{-j\theta/2}\left[e^{j\theta/2} - e^{-j\theta/2}\right]$$

$$x_k = \frac{1}{N} e^{jk\omega_0 N_1} \frac{e^{-jk\omega_0(N_1+1/2)}\left[e^{jk\omega_0(N_1+1/2)} - e^{-jk\omega_0(N_1+1/2)}\right]}{e^{-jk\omega_0/2}\left[e^{jk\omega_0/2} - e^{-jk\omega_0/2}\right]}$$