İşaretler ve Sistemler

Ders 8–10 – Sürekli zamanlı Fourier dönüşümü

Orijinal not: Isaret-ve-Sistemler-ders-8-10.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Sürekli Zamanlı Fourier Transformu (Dönüşümü)

orj. s. 1

(Continuous-Time Fourier Transform)

Periyodik olmayan (aperiyodik) işaretler için Fourier dönüşümü tanımlıdır.

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

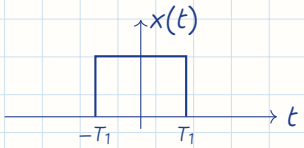
$$x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(f)$$

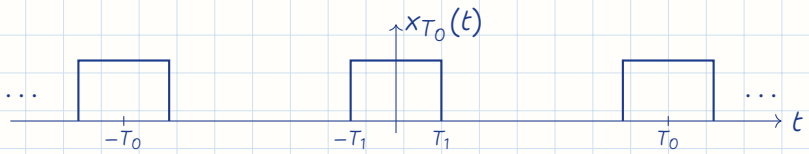
$\Rightarrow \omega$ notasyonu:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega, \qquad \omega = 2\pi f = 2\pi/T$$

$\Rightarrow$ Periyodik olmayan bir işaretin belli bir T_0 aralığında Fourier serisi açılımında T_0 sonsuza götürüldüğünde ($T_0 \to \infty$) Fourier transformu elde edilir.





$T_0 \to \infty$ iken $x_{T_0}(t) = x(t)$ dır (herhangi bir sonlu t değeri için)

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}, \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \underset{\uparrow}{=} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt \underset{\uparrow}{=} \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$x(t) = \begin{cases} x_{T_0}(t), & |t| < T_0/2 \\ 0, & |t| > T_0/2 \end{cases}$; $T_0 \to \infty$

orj. s. 2

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}$$

$$x_k = \frac{1}{T_0} X(kf_0) \qquad x_k = \frac{1}{T_0} X(k\omega_0)$$

$df = 1/T_0 = f_0$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(kf_0)\, e^{j2\pi kf_0 t} \qquad x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0)\, e^{jk\omega_0 t}$$

$$\Delta f = \frac{1}{T_0} \qquad \Delta\omega = \frac{2\pi}{T_0}$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} X(k\Delta f)\, e^{j2\pi k\Delta f t}\, \Delta f \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\Delta\omega)\, e^{jk\Delta\omega t}\, \Delta\omega$$

$$x(t) = \lim_{T_0\to\infty} x_{T_0}(t) = \underbrace{\lim_{\Delta f\to 0} \sum_{k=-\infty}^{\infty} X(k\Delta f)\, e^{j2\pi k\Delta f t}\, \Delta f}_{X(f)e^{j2\pi ft}\text{'nin altında kalan alan}} \qquad x(t) = \underbrace{\lim_{\Delta\omega\to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\Delta\omega)\, e^{jk\Delta\omega t}\, \Delta\omega}_{X(\omega)e^{j\omega t}}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df \qquad x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad [1]$$

⇒ Fourier dönüşümünün mevcut olma koşulları (Dirichlet condition)

1) Aperyodik $x(t)$ işareti mutlak integrallenebilir olmalıdır, $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$
2) Herhangi bir sonlu t aralığında,
   - sonlu sayıda maksimum ve minimum değere sahip olmalıdır.
   - sonlu sayıda süreksizliğe ve bu noktalarda sonlu değere sahip olmalıdır.

⇒ Frekans spektrumları:

$$X(f) = |X(f)|\, e^{j\theta_x(f)} \longrightarrow \begin{array}{l} |X(f)| : \text{ genlik spektrumu} \\ \hookrightarrow \theta_x(f) : \text{ faz spektrumu} \end{array}$$

Örnek

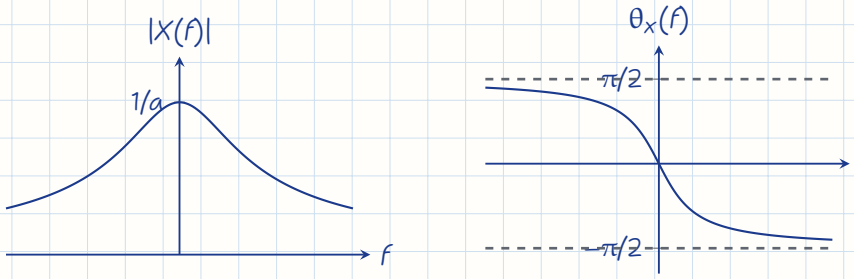
Ex: $x(t) = e^{-at}u(t)$, $a > 0$ ise $X(f) = ?$ Frekans spektrumlarını çiziniz.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_0^{\infty} e^{-at}\cdot e^{-j2\pi ft}\, dt = \int_0^{\infty} e^{-(a+j2\pi f)t}\, dt = \frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\Big|_0^{\infty} = \frac{1}{a+j2\pi f} = \frac{1}{a+j\omega}$$

$$|X(f)| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}}$$

[1] Claude notu: muhtemelen önünde $\frac{1}{2\pi}$ olmalı: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$.

$$\theta_x(f) = -\tan^{-1}(2\pi f/a)$$



Örnek

orj. s. 3

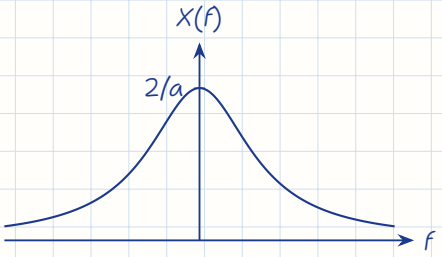
Ex: $x(t) = e^{-a|t|}$, $a > 0$ ise $X(f) = ?$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \underset{x(t)=\begin{cases} e^{at}, & t<0 \\ e^{-at}, & t>0 \end{cases}}{=} \int_{-\infty}^{0} e^{at} e^{-j2\pi ft}\, dt + \int_{0}^{\infty} e^{-at} e^{-j2\pi ft}\, dt$$

$$= \frac{e^{(a-j2\pi f)t}}{(a-j2\pi f)}\bigg|_{-\infty}^{0} + e^{-(a+j2\pi f)t}\bigg|_{0}^{\infty} = \frac{1}{a-j2\pi f} + \frac{1}{a+j2\pi f} = \frac{2a}{a^2+(2\pi f)^2}$$ [2]

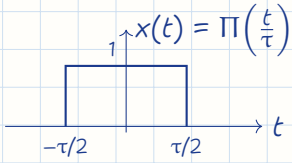
$$|X(f)| = \frac{2a}{a^2+(2\pi f)^2}$$

$$\theta_x(f) = 0$$



Örnek

Ex: $x(t) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & \text{dışında} \end{cases}$ dikdörtgen darbe işaretinin FT bulunuz.

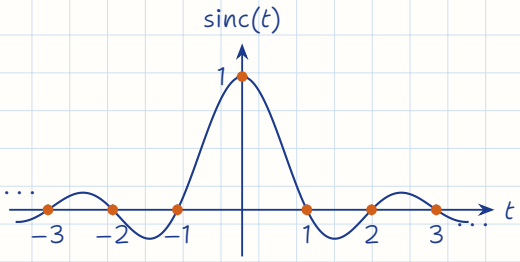


$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\, dt = \frac{e^{-j2\pi ft}}{-j2\pi f}\bigg|_{-\tau/2}^{\tau/2}$$

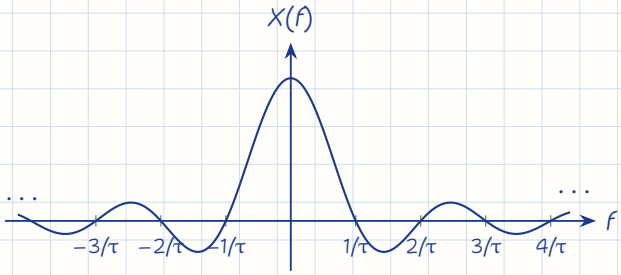
$$= \frac{-1}{j2\pi f}\left[e^{-j2\pi f\tau/2} - e^{+j2\pi f\tau/2}\right] = \frac{1}{\pi f}\sin(\pi f\tau)$$

Reminder: $\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

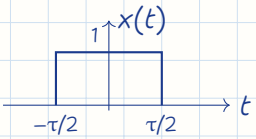
[2] Claude notu: ikinci terimde $-(a + j2\pi f)$ paydası unutulmuş; sonuç doğru.



$$X(f) = \frac{1}{\pi f}\sin(\pi f\tau) = \tau\,\frac{\sin(\pi f\tau)}{\pi f\tau} = \tau\,\mathrm{sinc}(f\tau)$$



$$x(t) = \Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \tau\,\mathrm{sinc}(f\tau)$$

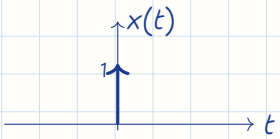


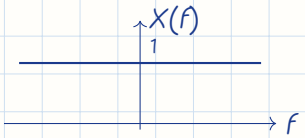
Örnek

Ex: $x(t) = \delta(t)$ ise $X(f) = ?$

$$X(f) = \int_{-\infty}^{\infty} \delta(t)\,e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} \delta(t)\,e^{-j2\pi f\cdot 0}\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$x(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = 1$$





Örnek

orj. s. 4

Ex: $X(f) = \delta(f - f_0)$ ise $x(t) = ?$

$$x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f-f_0)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f-f_0)\,e^{j2\pi f_0 t}\,df = e^{j2\pi f_0 t}\underbrace{\int_{-\infty}^{\infty} \delta(f-f_0)\,df}_{=1} = e^{j2\pi f_0 t}$$

Örnek

Ex: $X(f) = \delta(f)$ ise $x(t) = ?$

$$x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f)\,e^{j2\pi 0 t}\,df = \int_{-\infty}^{\infty} \delta(f)\,df = 1$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(f)$$

$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f - f_0)$$

$$e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f + f_0)$$

Periyodik işaretlerin Fourier Dönüşümü

$$x(t) = x(t+T),\ \forall t \qquad T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{jk\omega_0 t} \underset{\omega_0 = 2\pi f_0}{=} \sum_{k=-\infty}^{\infty} x_k\, e^{j2\pi k f_0 t} = \cdots + x_{-1}e^{-j2\pi f_0 t} + x_0 + x_1 e^{j2\pi f_0 t} + x_2 e^{j2\pi 2 f_0 t} + \cdots$$

$$x_1 e^{+j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} x_1\,\delta(f - f_0)$$

$$x_0 \overset{\mathcal{F}}{\longleftrightarrow} x_0\,\delta(f)$$

$$x_2 e^{j2\pi 2 f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} x_2\,\delta(f - 2f_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{j2\pi k f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \sum_{k=-\infty}^{\infty} x_k\,\delta(f - k f_0)$$

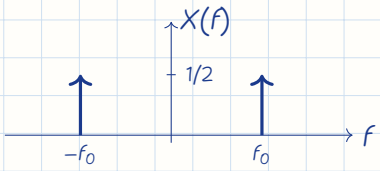
$$x_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

Örnek

Ex: $x(t) = \cos 2\pi f_0 t$ ise $X(f) = ?$

$$x(t) = \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] \longrightarrow x_1 = x_{-1} = 1/2$$

$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$



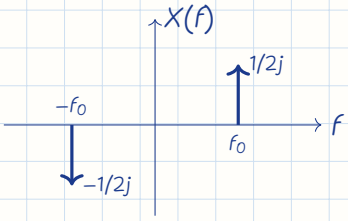
Örnek

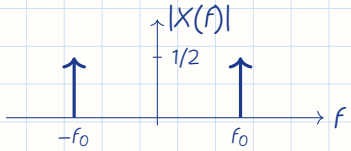
Ex: $x(t) = \sin 2\pi f_0 t$

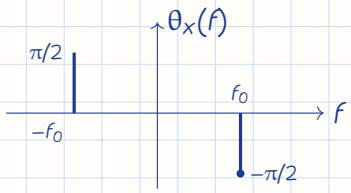
$$x(t) = \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \longrightarrow x_1 = \frac{1}{2j} = \frac{-1}{2}j$$

$$X(f) = \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right] \qquad x_{-1} = -\frac{1}{2j} = \frac{1}{2}j$$

$$|x_1| = |x_{-1}| = 1/2 \qquad \theta_{x_1} = -\pi/2 \quad \theta_{x_{-1}} = \pi/2$$



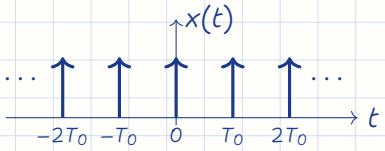




Örnek

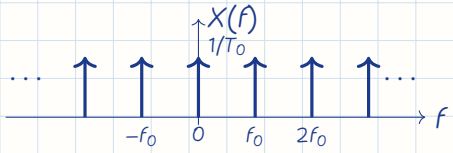
orj. s. 5

Ex: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$ dürtü katarı işaretinin FT bulunuz.



Reminder: $x_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} = \frac{1}{T_0}$

$$X(f) = \sum_{k=-\infty}^{\infty} x_k\, \delta(f - kf_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} \delta(f - kf_0)$$



Sürekli Zamanlı Fourier Transformunun Özellikleri

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$
$$X(f) = \mathcal{F}\{x(t)\}$$
, $x(t)$: aperiyodik işaret

1) Lineerlik: $\alpha x_1(t) + \beta x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X_1(f) + \beta X_2(f)$

2) Zamanda öteleme: $x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_0} X(f)$

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi ft}\, dt \underset{\uparrow\, t-t_0=\tau}{=} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(\tau+t_0)}\, d\tau = e^{-j2\pi ft_0} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\, d\tau}_{X(f)}$$

Aperiyodik bir $x(t)$ işareti zamanda ötelendiğinde genlik spektrumu değişmez, faz spektrumu $-2\pi ft_0$ kadar ötelenir.

3) Frekansta öteleme: $x(t)\, e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(f - f_0)$

İspat:

$$\mathcal{F}\left\{x(t)e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f-f_0)t}\, dt = X(f - f_0)$$

$x(t)$ işareti $e^{j2\pi f_0 t}$ ile çarpıldığında frekans spektrumu f_0 kadar ötelenir.

4) Modülasyon: $x(t)\cos 2\pi f_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$

$$x(t)\sin 2\pi f_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}\left[X(f - f_0) - X(f + f_0)\right] = \frac{1}{2}\left[X(f - f_0)\, e^{-j\pi/2} + X(f + f_0)\, e^{j\pi/2}\right]$$

5) Zamanda ölçeklendirme: $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X(f/a)$, $a \neq 0$ ⟶

$a > 1$ işaret frekansta (spektrum) genişler.

$0 < a < 1$ işaret frekansta daralır.

6) Duality: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$

$X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$

İspat:

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

$$x(-t) = \int_{-\infty}^{\infty} X(f)\, e^{-j2\pi ft}\, df \underset{\uparrow\, t\to f}{\Rightarrow} x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi ft}\, dt$$

orj. s. 6

7) Türev alma: $\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j2\pi f X(f)$

$$\frac{d^n x(t)}{dt^n} \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi f)^n X(f)$$

İspat:

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = \mathcal{F}\left\{\frac{d}{dt}\Big[\underbrace{\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, dt}_{x(t)=\mathcal{F}^{-1}\{X(f)\}}\Big]\right\} = \mathcal{F}\Big\{j2\pi f \underbrace{\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, dt}_{x(t)}\Big\}$$

$$= j2\pi f\, \mathcal{F}\{x(t)\} = j2\pi f X(f)$$ [3]

8) İntegral alma: $\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\,\delta(f)$

9) Konvolüsyon: $x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) \cdot X_2(f)$

10) Çarpma: $x_1(t) \cdot x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) * X_2(f)$

11) Eşlenik alma (conjugation): $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-f)$

$\Rightarrow$ $x(t)$ reel ise FT eşlenik simetriktir.

$$x(t) = x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = X^*(-f)$$

$\Rightarrow$ $x(t)$ reel ve çift ise FT reel ve çifttir.

$$\begin{matrix} x(t) = x^*(t) \\ x(t) = x(-t) \end{matrix} \overset{\mathcal{F}}{\longleftrightarrow} \begin{matrix} X^*(f) = X(f) \\ X(f) = X(-f) \end{matrix}$$

$\Rightarrow$ $x(t)$ reel ve tek ise FT sanal ve tektir.

$$\begin{matrix} x(t) = x^*(t) \\ x(t) = -x(-t) \end{matrix} \overset{\mathcal{F}}{\longleftrightarrow} \begin{matrix} X^*(f) = -X(f) \\ X(-f) = -X(f) \end{matrix}$$

Parseval teoremi:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df \qquad \longleftarrow\ |X(f)|^2 :\ \text{Enerji spektral yoğunluğu (energy spectral density)}$$

[3] Claude notu: integrallerdeki dt muhtemelen df olmalı.

İspat:

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} x(t)\,x^*(t)\,dt = \int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} X^*(f)\,e^{-j2\pi ft}\,df\right]dt$$
$$= \int_{-\infty}^{\infty} X^*(f)\Big[\underbrace{\int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt}_{X(f)}\Big]\,df$$
$$= \int_{-\infty}^{\infty} X^*(f)\,X(f)\,df = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

Örnek

Ex: $x(t) = A\,\Pi(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = A\tau\,\mathrm{sinc}(f\tau)$ ise $x(t) = \mathrm{sinc}(t\tau)$ işaretinin FT = ?

Reminder: $\left.\begin{array}{l} x(t) \longleftrightarrow X(f) \\ X(f) \longleftrightarrow x(-t) \end{array}\right\}$ Duality

$$x(t) = A\tau\,\mathrm{sinc}(t\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = A\,\Pi(-f/\tau) \underset{\text{Çift fonksiyon}}{=} A\,\Pi(f/\tau)$$

$$x(t) = \mathrm{sinc}(t\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \frac{1}{\tau}\,\Pi(f/\tau)$$

Örnek

orj. s. ?

Ex: $x(t) = \dfrac{2}{1+t^2}$ ise $X(\omega) = ?$

Reminder: $e^{-a|t|} \longleftrightarrow \dfrac{2a}{a^2 + (\omega)^2}$

$$x(t) = e^{-|t|} \longleftrightarrow X(\omega) = \frac{2}{1+(\omega)^2}$$

$$x(t) = \frac{2}{1+t^2} \longleftrightarrow X(\omega) = e^{-|-\omega|} = e^{-|\omega|}$$ [4]

Örnek

Ex: $\mathcal{F}\{u(t)\} = ?$

Reminder: $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\,\delta(f)$$

$$\mathcal{F}\{u(t)\} \underset{\substack{x(t)=\delta(t)\\ X(f)=1}}{=} \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Örnek

Ör: $x(t) = \dfrac{1}{1+t^2}$ Find the Fourier Transform.

[4] Claude notu: ω notasyonunda dualite $X(t) \leftrightarrow 2\pi x(-\omega)$ verir; muhtemelen $X(\omega) = 2\pi e^{-|\omega|}$ olmalı (aşağıdaki $\pi e^{-2\pi|f|}$ sonucuyla da uyumlu).

$$e^{-\alpha|t|} \longrightarrow \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \cdot \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

$$\left.\begin{array}{l} x(t) \longrightarrow X(f) \\ X(t) \longrightarrow x(-f) \end{array}\right\} \text{Duality}$$

$$X(t) = \frac{1}{1+t^2} \xrightarrow{\mathcal{F}} x(-f) \qquad \frac{2\alpha}{4\pi^2} \cdot \frac{1}{\frac{\alpha^2}{4\pi^2} + t^2} \xrightarrow{\mathcal{F}} e^{-\alpha|-f|} = e^{-\alpha|f|}$$

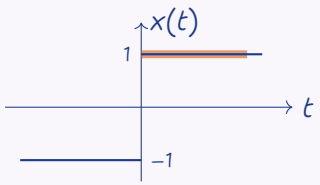
$$\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2} \longrightarrow \frac{4\pi^2}{2\alpha} \cdot e^{-\alpha|f|}$$

$$\frac{\alpha^2}{4\pi^2} = 1 \qquad \alpha^2 = 4\pi^2 \longrightarrow \underline{\alpha = 2\pi}$$

$$\frac{1}{1+t^2} \xrightarrow{\mathcal{F}} \pi e^{-2\pi|f|}$$

Örnek

Ex: $x(t) = \text{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$ ise $X(f) = ?$



$$x(t) = \text{sgn}(t) = u(t) - u(-t)$$

Reminder: $x(at) \longleftrightarrow \frac{1}{|a|} X(f/a)$

$$\mathcal{F}\{\text{sgn}(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) - \frac{1}{|-1|}\left[\frac{1}{j2\pi(-f)} + \frac{1}{2}\delta(-f)\right]$$

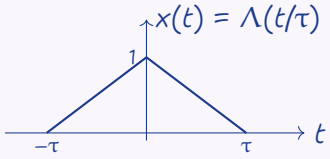
$$= \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} - \frac{1}{2}\delta(f) = \frac{1}{j\pi f}$$

$\uparrow$ $\delta(f)=\delta(-f)$ Çift fonksiyon

Örnek

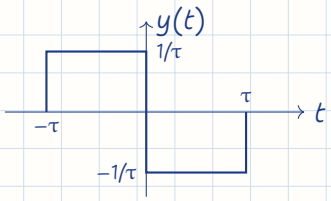
orj. s. 8

Ex: $x(t) = \Lambda(t/\tau)$ üçgen işaretinin FT bulunuz.



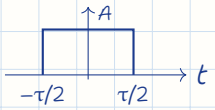
$$x(t) = \begin{cases} 1 + \frac{t}{\tau}, & -\tau < t < 0 \\ 1 - \frac{t}{\tau}, & 0 < t < \tau \end{cases}$$

$$y(t) = \frac{dx(t)}{dt} = \begin{cases} 1/\tau, & -\tau < t < 0 \\ -1/\tau, & 0 < t < \tau \end{cases}$$



Reminder: $\frac{dx(t)}{dt} \longleftrightarrow j2\pi f X(f) \Rightarrow Y(f) = j2\pi f X(f) \longrightarrow X(f) = \frac{1}{j2\pi f} Y(f)$

Reminder: $z(t) = A\,\Pi(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} Z(f) = A\tau\,\mathrm{sinc}(f\tau)$



$$A = 1/\tau \longrightarrow z(t) = \frac{1}{\tau}\Pi(t/\tau) \longleftrightarrow Z(f) = \mathrm{sinc}(f\tau)$$

$$y(t) = z(t + \tau/2) - z(t - \tau/2)$$

$$Y(f) = Z(f)\,e^{j2\pi f\tau/2} - Z(f)\,e^{-j2\pi f\tau/2}$$

$$X(f) = \frac{1}{j2\pi f} Y(f) = \frac{1}{j2\pi f} Z(f)\left[e^{j\pi f\tau} - e^{-j\pi f\tau}\right] = \frac{Z(f)}{\pi f}\sin(\pi f\tau) = Z(f)\,\tau\,\mathrm{sinc}(f\tau) = \tau\,\mathrm{sinc}^2(f\tau)$$

$$x(t) \triangleq \Lambda(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \triangleq \tau\,\mathrm{sinc}^2(f\tau)$$

10. Hafta

LTI Sistemlerin Frekans Domeni Analizi

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * \underset{\substack{\downarrow \\ \text{dürtü cevabı} \\ \text{(impulse response)}}}{h(t)} \overset{\mathcal{F}}{\longleftrightarrow} Y(f) = X(f) \underset{\substack{\downarrow \\ \text{frekans cevabı} \\ \text{(frequency response)}}}{H(f)}$$

$H(f) = |H(f)|\,e^{j\theta_h(f)} \longrightarrow |H(f)|$: genlik cevabı (amplitude response)

$\hookrightarrow \theta_h(f)$: faz cevabı (phase response)

$|Y(f)|\,e^{j\theta_y(f)} = |X(f)|\,e^{j\theta_x(f)}\,|H(f)|\,e^{j\theta_h(f)} \longrightarrow |Y(f)| = |X(f)|\,|H(f)|$

$\hookrightarrow \theta_y(f) = \theta_x(f) + \theta_h(f)$

Örnek

orj. s. 9

Ex: $x(t) = e^{-at}u(t),\ a > 0$ ise $y(t) = ?$

$h(t) = e^{-bt}u(t),\ b > 0$

$$y(t) = x(t) * h(t) = \mathcal{F}^{-1}\{Y(f) = X(f)H(f)\}$$

Reminder: $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j2\pi f}$

$$Y(f) = \frac{1}{(a + j2\pi f)(b + j2\pi f)} = \frac{A}{a + j2\pi f} + \frac{B}{b + j2\pi f} \quad \longleftarrow A = -B = \frac{1}{b - a}$$

$$Y(f) = \frac{1}{b - a}\left[\frac{1}{a + j2\pi f} - \frac{1}{b + j2\pi f}\right] \overset{\mathcal{F}^{-1}}{\longleftrightarrow} y(t) = \frac{1}{b - a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

Örnek

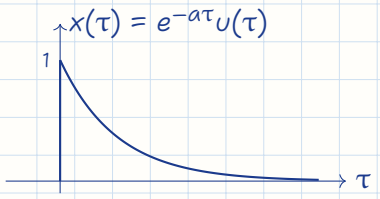
Ex: $x(t) = h(t) = e^{-at}u(t)$, $a > 0$ ise $y(t) = ?$

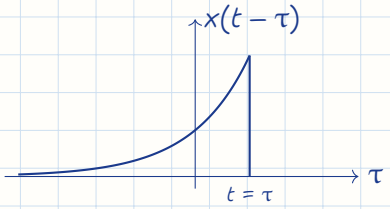
$$Y(f) = \left(\frac{1}{a + j2\pi f}\right)^2$$

Reminder: $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j2\pi f}$

$$x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) \cdot X_2(f)$$

$$y(t) = \mathcal{F}^{-1}\{X(f)X(f)\} = x(t) * x(t) = e^{-at}u(t) * e^{-at}u(t)$$





$$y(t) = x(t) * x(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)\, e^{-a(t-\tau)}u(t-\tau)\, d\tau = e^{-at}\int_0^t 1\, d\tau = t\, e^{-at}, \qquad t > 0$$

$$t\, e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(a + j2\pi f)^2}$$

Örnek

Ex: $x(t) = e^{-2t}u(t)$, $y(t) = e^{-t}u(t)$ ise sistemin frekans cevabını ve dürtü cevabını bulunuz.

$$Y(f) = X(f)H(f) \longrightarrow H(f) = \frac{Y(f)}{X(f)} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} h(t)$$

$$H(f) = \frac{\frac{1}{1 + j2\pi f}}{\frac{1}{2 + j2\pi f}} = \frac{2 + j2\pi f}{1 + j2\pi f} = 1 + \frac{1}{1 + j2\pi f} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} h(t) = \delta(t) + e^{-t}u(t)$$

Reminder: $1 \longleftrightarrow \delta(f)$

$\delta(t) \longleftrightarrow 1$

Örnek

orj. s. 10

Ex: $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$ ile tanımlanan LTI sistemin dürtü cevabını bulunuz.

$$(j2\pi f)^2 Y(f) + 6j2\pi f Y(f) + 8Y(f) = 2X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{2}{-(2\pi f)^2 + 6j(2\pi f) + 8} \underset{\omega = 2\pi f}{=} \frac{2}{-\omega^2 + 6j\omega + 8} = \frac{2}{(j\omega + 4)(j\omega + 2)} = \frac{A}{j\omega + 4} + \frac{B}{j\omega + 2}$$

$j\omega \quad 4$

$j\omega \quad 2$

$$A = -B = -1$$

$$H(f) = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 4} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} h(t) = \left(e^{-2t} - e^{-4t}\right)u(t)$$