İşaretler ve Sistemler

Ders 8 – Sürekli zamanlı Fourier dönüşümü

Orijinal not: Isaret-ve-Sistemler-ders-8.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

Sürekli Zamanlı Fourier Transformu (Dönüşümü)

orj. s. 1

(Continuous-Time Fourier Transform)

Periyodik olmayan (aperiyodik) işaretler için Fourier dönüşümü tanımlıdır.

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

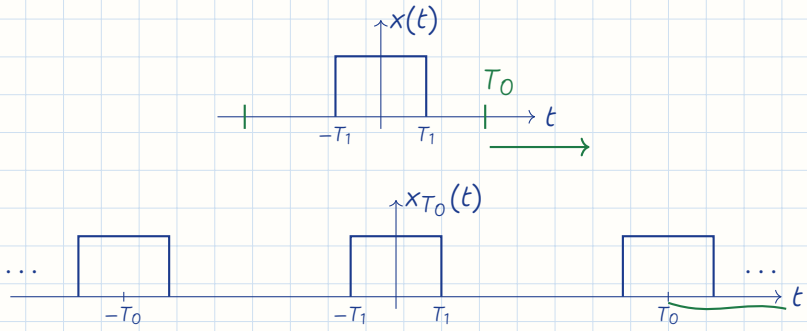
$$x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(f)$$

$\Rightarrow$ ω notasyonu:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega, \qquad \omega = 2\pi f = 2\pi/T$$

$\Rightarrow$ Periyodik olmayan bir işaretin belli bir T_0 aralığında Fourier serisi açılımında T_0 sonsuza götürüldüğünde ($T_0 \to \infty$) Fourier transformu elde edilir.



$T_0 \to \infty$ iken $x_{T_0}(t) = x(t)$ dır (herhangi bir sonlu t değeri için)

$$\longrightarrow \lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}, \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \underset{\uparrow}{=} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt \underset{\uparrow}{=} \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$x(t) = \begin{cases} x_{T_0}(t), & |t| < T_0/2 \\ 0, & |t| > T_0/2 \end{cases} \qquad T_0 \to \infty$$

orj. s. 2

$$\longrightarrow X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}$$

$$\longrightarrow x_k = \frac{1}{T_0} X(kf_0) \longleftarrow \qquad x_k = \frac{1}{T_0} X(k\omega_0)$$

$df = 1/T_0 = f_0$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(kf_0)\, e^{j2\pi kf_0 t} \qquad x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0)\, e^{jk\omega_0 t}$$

$$\longrightarrow \Delta f = \frac{1}{T_0} \qquad \Delta\omega = \frac{2\pi}{T_0}$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} X(k\Delta f)\, e^{j2\pi k\Delta f t}\, \Delta f \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\Delta\omega)\, e^{jk\Delta\omega t}\, \Delta\omega$$

$$x(t) = \lim_{T_0\to\infty} x_{T_0}(t) \qquad x(t) = \underbrace{\lim_{\Delta\omega\to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\Delta\omega)\, e^{jk\Delta\omega t}\, \Delta\omega}_{X(\omega)e^{j\omega t}}$$

$$= \underbrace{\lim_{\Delta f\to 0} \sum_{k=-\infty}^{\infty} X(k\Delta f)\, e^{j2\pi k\Delta f t}\, \Delta f}_{X(f)e^{j2\pi ft}\text{'nin altında kalan alan}} \downarrow$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad [1]$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

⇒ Fourier dönüşümünün mevcut olma koşulları (Dirichlet condition)

1) Aperyodik $x(t)$ işareti mutlak integrallenebilir olmalıdır, $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

2) Herhangi bir sonlu t aralığında,
   - sonlu sayıda maksimum ve minimum değere sahip olmalıdır.
   - sonlu sayıda süreksizliğe ve bu noktalarda sonlu değere sahip olmalıdır.

⇒ Frekans spektrumları:

$$\nearrow X(f) = |X(f)|\, e^{j\theta_x(f)} \longrightarrow |X(f)| : \text{genlik spektrumu}$$
$$\hookrightarrow \theta_x(f) : \text{faz spektrumu}$$

orj. s. 3

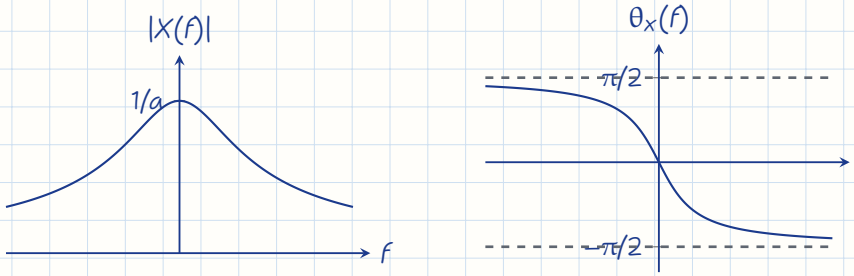
Örnek

Ex: $x(t) = e^{-at}u(t)$, $a > 0$ ise $X(f) = ?$ Frekans spektrumlarını çiziniz.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_0^{\infty} e^{-at}\cdot e^{-j2\pi ft}\, dt = \int_0^{\infty} e^{-(a+j2\pi f)t}\, dt = \frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\Big|_0^{\infty} = \frac{1}{a+j2\pi f} = \frac{1}{a+j\omega}$$

[1] Claude notu: muhtemelen önünde $\frac{1}{2\pi}$ olmalı: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$.

$$|X(f)| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}}$$
$$\theta_x(f) = -\tan^{-1}(2\pi f/a)$$



Örnek

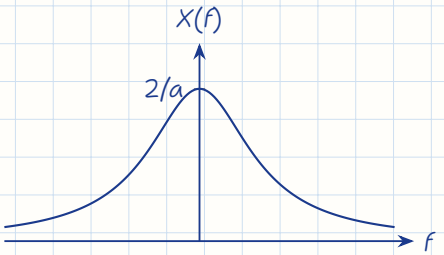
Ex: $x(t) = e^{-a|t|}$, $a > 0$ ise $X(f) = ?$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \underset{\uparrow}{=} \int_{-\infty}^{0} e^{at} e^{-j2\pi ft}\, dt + \int_{0}^{\infty} e^{-at} e^{-j2\pi ft}\, dt$$

$$x(t) = \begin{cases} e^{at}, & t<0 \\ e^{-at}, & t>0 \end{cases}$$

$$= \frac{e^{(a-j2\pi f)t}}{(a-j2\pi f)}\Bigg|_{-\infty}^{0} + e^{-(a+j2\pi f)t}\Big|_{0}^{\infty} = \frac{1}{a-j2\pi f} + \frac{1}{a+j2\pi f} = \frac{2a}{a^2+(2\pi f)^2}$$ [2]

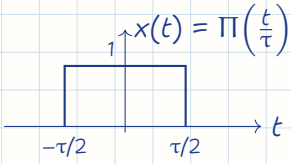
$$|X(f)| = \frac{2a}{a^2 + (2\pi f)^2}$$
$$\theta_x(f) = 0$$



orj. s. 4

Örnek

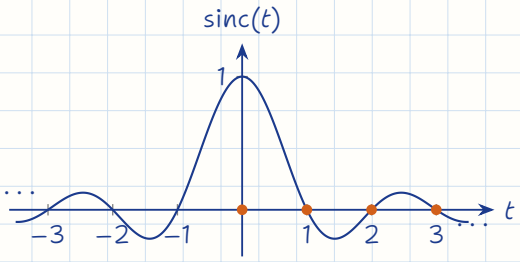
Ex: $x(t) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & \text{dışında} \end{cases}$ dikdörtgen darbe işaretinin FT bulunuz.



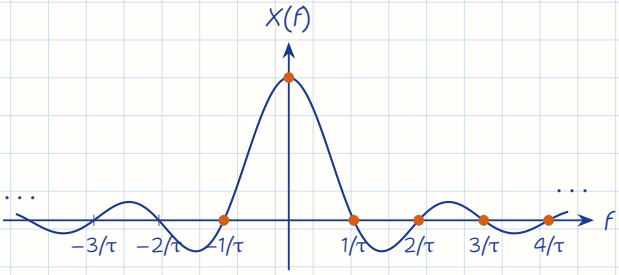
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\, dt = \frac{e^{-j2\pi ft}}{-j2\pi f}\Bigg|_{-\tau/2}^{\tau/2}$$
$$= \frac{-1}{j2\pi f}\left[e^{-j2\pi f\tau/2} - e^{+j2\pi f\tau/2}\right] = \frac{1}{\pi f}\sin(\pi f\tau)$$

Reminder: $\text{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

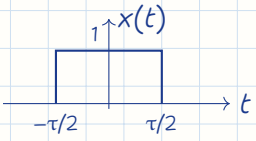
[2] Claude notu: ikinci terimde $-(a + j2\pi f)$ paydası unutulmuş; sonuç doğru.



$$X(f) = \frac{1}{\pi f}\sin(\pi f\tau) = \tau\,\frac{\sin(\pi f\tau)}{\pi f\tau} = \tau\,\mathrm{sinc}(f\tau)$$



$$x(t) = A\,\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = A\,\tau\,\mathrm{sinc}(f\tau)$$



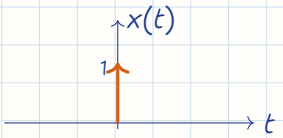
orj. s. 5

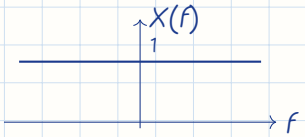
Örnek

Ex: $x(t) = \delta(t)$ ise $X(f) = ?$

$$X(f) = \int_{-\infty}^{\infty} \delta(t)\,e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} \delta(t)\,\underbrace{e^{-j2\pi f\cdot 0}}_{1}\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$x(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = 1$$





Örnek

Ex: $X(f) = \delta(f - f_0)$ ise $x(t) = ?$

$$x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f-f_0)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f-f_0)\,e^{j2\pi f_0 t}\,df = e^{j2\pi f_0 t}\underbrace{\int_{-\infty}^{\infty} \delta(f-f_0)\,df}_{=1} = e^{j2\pi f_0 t}$$

Örnek

Ex: $X(f) = \delta(f)$ ise $x(t) = ?$

$$x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f)\,e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} \delta(f)\,e^{j2\pi 0 t}\,df = \int_{-\infty}^{\infty} \delta(f)\,df = 1$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(f)$$

$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f - f_0)$$

$$e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f + f_0)$$

orj. s. 6

Periyodik işaretlerin Fourier Dönüşümü

$$x(t) = x(t+T),\ \forall t \qquad T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} x_k\, e^{j2\pi k f_0 t} = \cdots + x_{-1}e^{-j2\pi f_0 t} + x_0 + x_1 e^{j2\pi f_0 t} + x_2 e^{j2\pi 2 f_0 t} + \cdots$$

($\omega_0 = 2\pi f_0$)

$$x_1 e^{+j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} x_1\,\delta(f - f_0)$$

$$x_0 \overset{\mathcal{F}}{\longleftrightarrow} x_0\,\delta(f)$$

$$x_2 e^{j2\pi 2 f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} x_2\,\delta(f - 2f_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k\, e^{j2\pi k f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \sum_{k=-\infty}^{\infty} x_k\,\delta(f - k f_0)$$

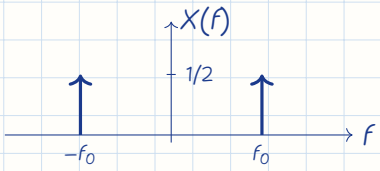
$$x_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

Örnek

Ex: $x(t) = \cos 2\pi f_0 t$ ise $X(f) = ?$

$$x(t) = \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] \longrightarrow x_1 = x_{-1} = 1/2$$

$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$



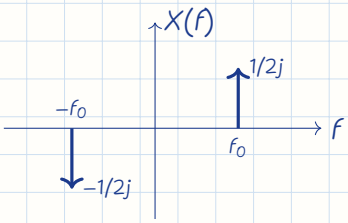
Örnek

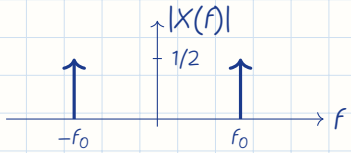
Ex: $x(t) = \sin 2\pi f_0 t$

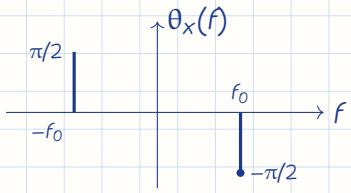
$$x(t) = \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \longrightarrow x_1 = \frac{1}{2j} = \frac{-1}{2}j$$

$$X(f) = \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right] \qquad x_{-1} = -\frac{1}{2j} = \frac{1}{2}j$$

$$|x_1| = |x_{-1}| = 1/2 \qquad \theta_{x_1} = -\pi/2 \quad \theta_{x_{-1}} = \pi/2$$



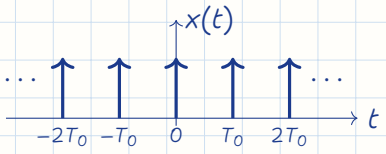




orj. s. 7

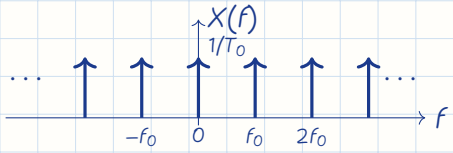
Örnek

Ex: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$ dürtü katarı işaretinin FT bulunuz.



Reminder: $x_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} = \frac{1}{T_0}$

$$X(f) = \sum_{k=-\infty}^{\infty} x_k\, \delta(f - kf_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} \delta(f - kf_0)$$





orijinal çizim, s. 7

Sürekli Zamanlı Fourier Transformunun Özellikleri

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$
$$X(f) = \mathcal{F}\{x(t)\}$$
, $x(t)$: aperiyodik işaret

1) Lineerlik: $\alpha x_1(t) + \beta x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X_1(f) + \beta X_2(f)$

2) Zamanda öteleme: $x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_0} X(f)$

$$\mathcal{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j2\pi ft}\, dt \underset{\uparrow\, t - t_0 = \tau}{=} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(\tau + t_0)}\, d\tau = e^{-j2\pi f t_0} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\, d\tau}_{X(f)}$$

Aperiyodik bir $x(t)$ işareti zamanda ötelendiğinde genlik spektrumu değişmez, faz spektrumu $-2\pi f t_0$ kadar ötelenir.

3) Frekansta öteleme: $x(t)\, e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(f - f_0)$

İspat:

$$\mathcal{F}\left\{x(t) e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi (f - f_0)t}\, dt = X(f - f_0)$$

$x(t)$ işareti $e^{j2\pi f_0 t}$ ile çarpıldığında frekans spektrumu f_0 kadar ötelenir.

4) Modülasyon: $x(t)\cos 2\pi f_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(f-f_0)+X(f+f_0)\right]$

$$x(t)\sin 2\pi f_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}\left[X(f-f_0)-X(f+f_0)\right] = \frac{1}{2}\left[X(f-f_0)\,e^{-j\pi/2}+X(f+f_0)\,e^{j\pi/2}\right]$$

orj. s. 8

5) Zamanda ölçeklendirme: $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X(f/a),\ a \neq 0 \longrightarrow$

$a > 1$ işaret frekansta (spektrum) genişler.

$0 < a < 1$ işaret frekansta daralır.

6) Duality: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$

$X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$

İspat:

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df$$

$$x(-t) = \int_{-\infty}^{\infty} X(f)\,e^{-j2\pi ft}\,df \underset{t\to f}{\Rightarrow} x(-f) = \boxed{\int_{-\infty}^{\infty} X(t)\,e^{-j2\pi ft}\,dt} \qquad X(t)$$

7) Türev alma: $\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j2\pi f X(f)$

Örnek

Ex: $x(t) = A\,\Pi(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = A\tau\,\mathrm{sinc}(f\tau)$ ise $x(t) = \mathrm{sinc}(t\tau)$ işaretinin FT = ?

Reminder: $\left.\begin{array}{l} x(t) \longleftrightarrow X(f) \\ X(f) \longleftrightarrow x(-t) \end{array}\right\}$ Duality

$$x(t) = A\tau\,\mathrm{sinc}(t\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = A\,\Pi(-f/\tau) \underset{\text{Çift fonksiyon}}{=} A\,\Pi(f/\tau)$$

$$x(t) = \mathrm{sinc}(t\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \frac{1}{\tau}\Pi(f/\tau)$$



orijinal çizim, s. 8

Örnek

Ex: $x(t) = \frac{2}{1+t^2}$ ise $X(f)$ = ?

Reminder: $e^{-a|t|} \longleftrightarrow \dfrac{2a}{a^2 + (2\pi f)^2}$

$$x(t) = e^{-|t|} \longleftrightarrow X(f) = \frac{2}{1 + (2\pi f)^2}$$

$$x(t) = \frac{2}{1 + t^2} \longleftrightarrow X(f) = e^{-|-f|} = e^{-|f|}$$ [3]

orj. s. 9

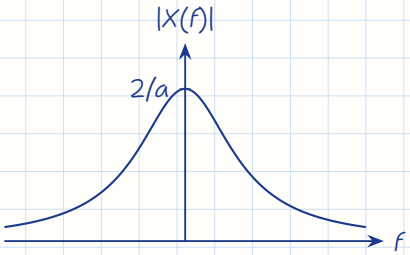
Örnek

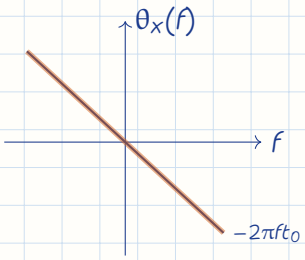
Ex: $x(t) = e^{-|t-t_0|}$ ise $X(f) = ?$

Reminder: $x(t - t_0) \longleftrightarrow e^{-j2\pi f t_0} X(f)$ ve $e^{-|t|} \longleftrightarrow \dfrac{2a}{a^2 + (2\pi f)^2}$

$$x(t) = e^{-|t-t_0|} \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \frac{2a}{a^2 + (2\pi f)^2} e^{-j2\pi f t_0} \longrightarrow |X(f)| = \frac{2a}{a^2 + (2\pi f)^2}$$

$$\hookrightarrow \theta_x(f) = -2\pi f t_0$$





[3] Claude notu: $X(f) = \frac{2}{1+(2\pi f)^2}$ olduğundan dualite $\frac{2}{1+(2\pi t)^2} \leftrightarrow e^{-|f|}$ verir; $\frac{2}{1+t^2}$ için sonuç muhtemelen $2\pi e^{-2\pi|f|}$ olmalı.