İşaretler ve Sistemler

13. hafta – Laplace dönüşümü, ROC, ters Laplace

Orijinal not: Sinyaller-ve-sistemler-13hafta.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1

Laplace transformu

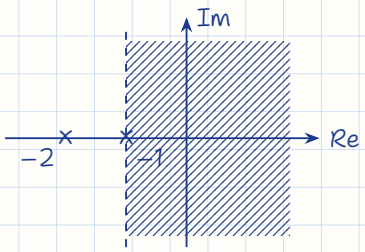
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Örnek

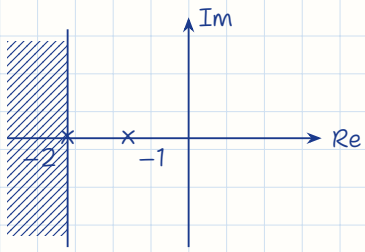
$X(s) = \dfrac{1}{(s+1)(s+2)}$ tüm olası ROC'lar = ?

$$= \frac{A}{s+1} + \frac{B}{s+2} = \underset{s=-1}{\frac{1}{s+1}} - \underset{s=-2}{\frac{1}{s+2}}$$

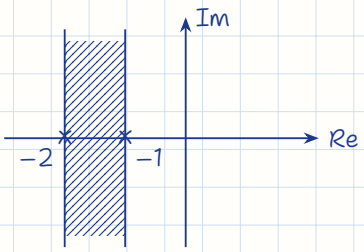
$$e^{-at}u(t) \rightarrow \frac{1}{s+a}, \quad \mathrm{Re}(s) > -a \qquad -e^{-at}u(-t) \leftrightarrow \frac{1}{s+a}$$



$x(t)$ sağ yanlı ise
$x(t) = e^{-t}u(t) - e^{-2t}u(t)$
ROC, $s = j\omega$ içerir, FT yakınsar



$x(t)$ sol yanlı ise
$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$
FT yoktur



$x(t)$ çift yanlı ise
$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$
FT yoktur

Ters Laplace Transform:

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-st}\, dt = \mathcal{F}\left\{x(t)\, e^{-\sigma t}\right\}^{a}$$

[a] Claude notu: muhtemelen $e^{-j\omega t}$ olmalı.

$$x(t)\, e^{-\sigma t} = \mathcal{F}^{-1}\left\{X(\sigma + j\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

$$s = \sigma + j\omega \;\rightarrow\; ds = j\, d\omega$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

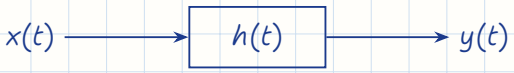
$$\rightarrow X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i} \longrightarrow \begin{array}{l} x(t) = A_i\, e^{-a_i t}\, u(t) \;:\; \text{sağ yanlı}, \\ \hookrightarrow x(t) = -A_i\, e^{-a_i t}\, u(-t) \;:\; \text{sol yanlı} \end{array}$$

orj. s. 2

Laplace Transformun özellikleri

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{ROC}: R$$

1) Lineerlik: $\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X_1(s) + \beta X_2(s)$, ROC: $R_1 \cap R_2$
2) Zamanda öteleme: $x(t - t_0) \longleftrightarrow e^{-st_0} X(s)$, ROC: R
3) S-domaininde öteleme: $e^{s_0 t} x(t) \longleftrightarrow X(s - s_0)$, ROC: $R + \text{Re}\{s_0\}$
   $s_0 = j\omega_0 \Rightarrow e^{j\omega_0 t} x(t) \longleftrightarrow X(s - j\omega_0)$, ROC: R
4) Zamanda ölçeklendirme: $x(at) = \frac{1}{|a|} X(s/a)$, ROC: R/a
   $a = -1 \rightarrow x(-t) \longleftrightarrow X(-s)$, ROC: $-R$
5) Eşlenik alma: $x^*(t) \longleftrightarrow X^*(s^*)$, ROC: R
6) Türev: $\frac{dx(t)}{dt} \longleftrightarrow sX(s)$, ROC: R'yi içerir
   $-t\, x(t) \longleftrightarrow \frac{dX(s)}{ds}$, ROC: R
7) İntegral: $\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{s} X(s)$, ROC: $R \cap \{\text{Re}\{s\} > 0\}$
8) Konvolüsyon: $x_1(t) * x_2(t) \longleftrightarrow X_1(s) X_2(s)$, ROC: $R_1 \cap R_2$



$$y(t) = x(t) * h(t) \longleftrightarrow Y(s) = X(s)H(s)$$

$$H(s) = \frac{Y(s)}{X(s)} \;:\; \text{Transfer fonksiyon}$$

Örnek

$x(t) = t\,e^{-at}u(t),\ X(s) = ?$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a},\quad \mathrm{Re}\{s\} > -a$$

orj. s. 3

$$t\,e^{-at}u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2},\quad \mathrm{Re}\{s\} > -a$$

$$\frac{t^2}{2}e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^3},\quad \mathrm{Re}\{s\} > -a$$

$$\vdots$$

$$\star\ \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^n},\quad \mathrm{Re}\{s\} > -a$$

Örnek

$H(s) = \dfrac{2s^2+5s+5}{(s+1)^2(s+2)},\ \mathrm{Re}\{s\} > -1,\ h(t) = ?$

$$H(s) = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2} \quad \rightarrow \quad \begin{matrix} A = 2 \\ B = -1 \\ C = 3 \end{matrix}$$

$$\updownarrow \mathcal{L}^{-1}$$

$$h(t) = 2t\,e^{-t}u(t) - e^{-t}u(t) + 3e^{-2t}u(t)$$

ROC ile LTI sistemlerinin özelliklerinin analizi

1) Nedensellik: $h(t) = 0,\ t < 0$ sağ yanlı → ROC sağ yanlı

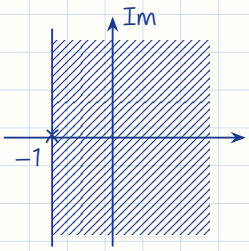
2) kararlılık: $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$ → ROC jω eksenini içermelidir.

Örnek

$h(t) = e^{-t}u(t)$, kararlı mıdır?

$$\updownarrow$$

$$H(s) = \frac{1}{s+1},\quad \mathrm{Re}\{s\} > -1$$



ROC sağ yanlı olduğundan nedenseldir !
jω'yi kesmektedir onun için kararlı!

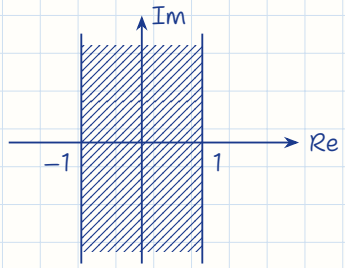
orj. s. 4

$$x(t) = e^{-b|t|}$$

$$X(s) = \frac{-2b}{s^2 - b^2},\quad -b < \mathrm{Re}\{s\} < b$$

Örnek

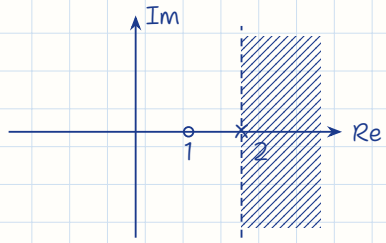
$$h(t) = e^{-|t|} \longleftrightarrow H(s) = \frac{-2}{s^2 - 1}, \quad -1 < \mathrm{Re}\{s\} < 1$$



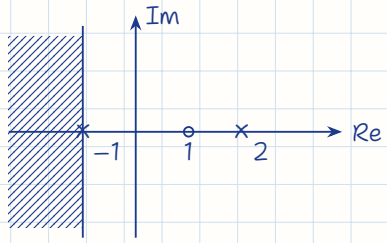
Nedensel değildir
kararlıdır

Örnek

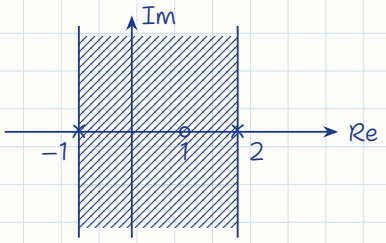
$$H(s) = \frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}, \quad A = 2/3, \quad B = \frac{1}{3}$$



$h(t)$ sağ yanlı
$h(t) = \frac{2}{3}e^{-t}u(t)$
$+\frac{1}{3}e^{2t}u(t)$
Nedensel (sağ yanlı)
kararlı değil (jω eksenini kesmez)



$h(t)$ sol yanlı
$h(t) = \frac{-2}{3}e^{-t}u(-t)$
$-\frac{1}{3}e^{2t}u(-t)$
Nedensel değil Sol yanlı
kararlı değil



$h(t)$ çift yanlı
$h(t) = \frac{2}{3}e^{-t}u(t)$
$-\frac{1}{3}e^{2t}u(-t)$
Nedensel değil
kararlı

orj. s. 5

Örnek

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t), \quad h(t) = ?$$

$$\updownarrow \mathcal{L}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) + 3X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2}, \qquad A = 2, \quad B = -1$$

$$\updownarrow \mathcal{L}^{-1}$$

$$h(t) = 2e^{-t}u(t) - e^{-t}u(t), \quad \mathrm{Re}\{s\} > -1\ ^{a}$$

[a] Claude notu: muhtemelen $-e^{-2t}u(t)$ olmalı ($B/(s+2)$ terimi için).

z transformu

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$x[n] \overset{z}{\longleftrightarrow} X(z)$$

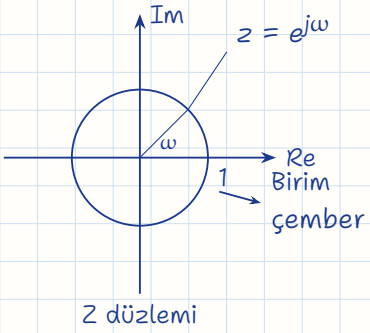
$$X(z) = \mathcal{Z}\{x[n]\}$$

$$\Rightarrow z = re^{j\omega} \quad \begin{cases} \longrightarrow r: \text{magnitude} \\ \hookrightarrow \omega: \text{angle} \end{cases}$$

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$
$$= \mathcal{F}\{x[n] r^{-n}\}$$

$$\Rightarrow z = e^{j\omega} \ (r = 1)$$

$$X(z) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \mathcal{F}\{x[n]\} = X(z)\Big|_{z=e^{j\omega}}$$



orj. s. 6

→ Yakınsama bölgesi (region of convergence): z-transform toplamının yakınsadığı z değer aralığıdır.

Not: z-transformun ROC'u birim çemberi içerirse Fourier transformu yakınsar.

Örnek

$a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

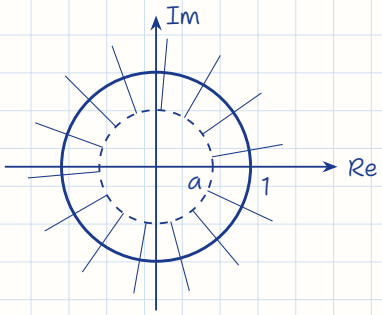
$$\text{Reminder}: \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |az^{-1}| < 1, \quad |z| > |a|$$

$z = 0$ sıfırı
$z = a$ kutupu

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a| \quad \swarrow \ a = 1$$

$$u[n] \longleftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$$



z düzlemi: / kutup-sıfır çizimi

x: kutup
o: sıfır
$0 < a < 1$

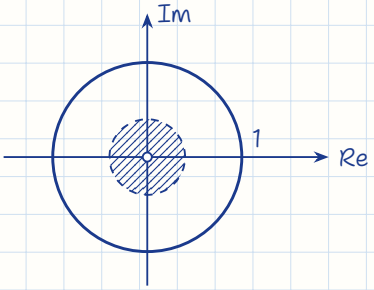
Örnek

$x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$$
$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$
$$= 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |a^{-1}z| < 1, \quad |z| < |a|$$

orj. s. 7

$$-a^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$$



$0 < a < 1$

$X(z)$'in ROC'u birim çemberi içermediğinden, $x[n]$'in Fourier Transformu yoktur.

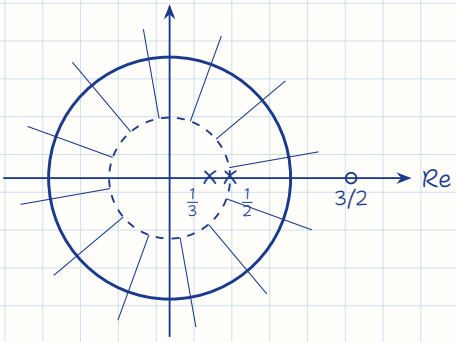
Örnek

$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

Reminder: $a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$

$$X(z) = \underbrace{\frac{7}{1-\frac{1}{3}z^{-1}}}_{\left|\frac{1}{3}z^{-1}\right| < 1 \;\Rightarrow\; |z| > \frac{1}{3}} - \underbrace{\frac{6}{1-\frac{1}{2}z^{-1}}}_{\left|\frac{1}{2}z^{-1}\right| < 1 \;\Rightarrow\; |z| > 1/2}$$

$$= \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{z(z-3/2)}{(z-1/3)(z-1/2)}, \quad |z| > 1/2$$



orj. s. 8

Örnek

$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi n}{4}\right) u[n], \quad X(z) = ?$

$$x[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]$$

$$X(z) = \frac{1}{2j}\underbrace{\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}}}_{\left|\frac{1}{3}e^{j\pi/4}z^{-1}\right|<1} - \frac{1}{2j}\underbrace{\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}}_{\left|\frac{1}{3}e^{-j\pi/4}z^{-1}\right|<1}$$

$$= \frac{z/(3\sqrt{2})}{\left(z-\frac{1}{3}e^{j\pi/4}\right)\left(z-\frac{1}{3}e^{-j\pi/4}\right)}, \quad |z| > 1/3$$

Not: Payın derecesi > paydanın derecesi → sonsuzda kutup vardır

< → sonsuzda sıfır vardır.

Yakınsaklık bölgesi (ROC) özellikleri

1) $X(z)$'nin ROC'u z düzleminde
2) ROC, herhangi bir kutup içermez
3) $x[n]$ sonlu süreli ise, ROC, olası $z = 0$ ve/veya $z = \infty$ değerleri dışında bütün z-düzlemidir.

Örnek

$x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = 1,$$

$\delta[n] = 1, \; n = 0$

orj. s. 9

$\delta[n] \overset{z}{\longleftrightarrow} 1, \quad \forall z$ (ROC, $z = 0$ ve $z = \infty$ içeren bütün z düzlemidir)

Örnek

$x[n] = \delta[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-1]z^{-n} = z^{-1} = \frac{1}{z}$$

$\delta[n-1]=1,\ n=1$

$z = 0$'da bir kutup var
ROC, $z = \infty$ içeren, $z = 0$ dışında bütün z düzlemidir.

Örnek

$x[n] = \delta[n+1]$

$$X(z) = \delta[n+1]^{a}$$

[a] Claude notu: muhtemelen $x[n] = \delta[n+1]$ olmalı.

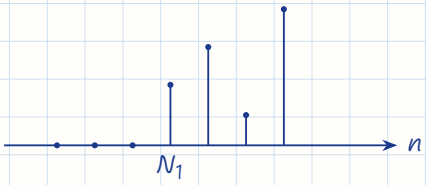
$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1]z^{-n} = z$$

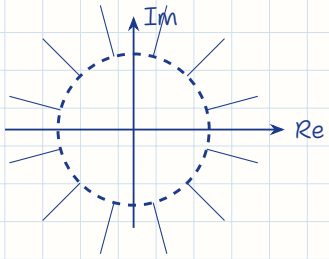
$\uparrow$

$\delta[n+1] = 1,\ n = -1$

$z = \infty$'da bir kutup vardır
ROC, $z = 0$ içeren, $z = \infty$ dışında bütün z düzlemidir.

4) $x[n]$ sağ yanlı ve $|z| = r_0$ çemberi ROC'da ise, $|z| > r_0$ sağlayan sonlu bütün z değerleri de ROC'dadır.





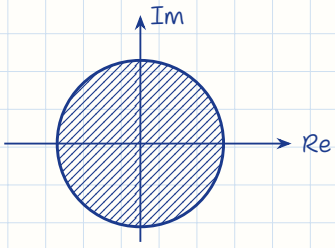
$$X(z) = \sum_{n=}^{\infty} x[n]z^{-n} \rightarrow$$ $N_1 < 0$ iken $z \to \infty$, $X(z)$ sonsuzdur ve ROC, $z = \infty$ içermez[a]

[a] Claude notu: toplamın alt sınırı yazılmamış (muhtemelen $n = N_1$).

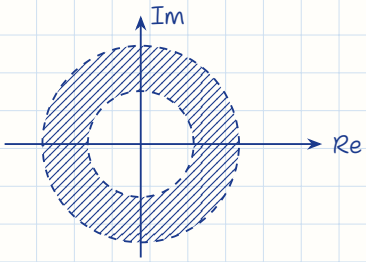
orj. s. 10

5) $x[n]$ sol yanlı ve $|z| = r_0$ çemberi ROC'da ise, $0 < |z| < r_0$ sağlayan bütün z değerleri de ROC'da olur





6) $x[n]$ iki yanlı ve $|z| = r_0$ çemberi ROC'da ise, ROC, z-düzleminde $|z| = r_0$ çemberini içeren bir halkadır



7) $X(z)$ rasyonel ise, ROC, ya kutuplarla sınırlıdır ya da sonsuza genişler ve kutup içermez

→ $x[n]$ sağ yanlı ise, ROC, en dışındaki kutupların dışındaki z düzlemidir.

→ $x[n]$ sol yanlı ise, ROC, en içerdeki sıfırdan farklı kutupların içindeki z düzlemidir.