Kontrol Sistemleri

Root Locus

Original notes: Control-Systems-lecture-10.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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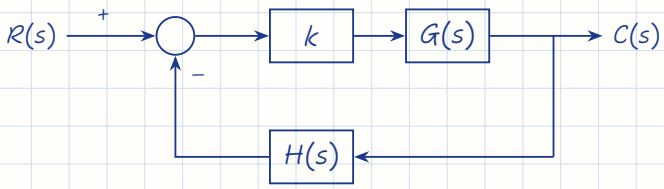
Root locus

All possible root locations of a system

orig. p. 1

→ For higher order systems the "degree" of polynomial in the characteristic equation is high!

↪ Finding root becomes difficult



accuracy → TF + $R(s)$

$$\frac{C(s)}{R(s)} = \frac{k\,G(s)}{1 + k\,G(s)H(s)}$$

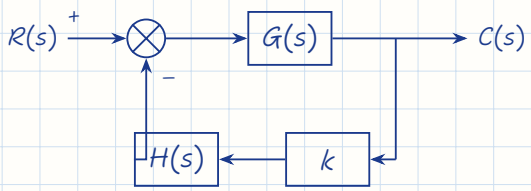
stability → $1 + k\,G(s)H(s) = 0$

Quality of response (t_s, t_p, M)

$$G(s) = \frac{N_G(s)}{D_G(s)}$$

$N_G(s)$ → Numerator, $D_G(s)$ → Denominator

$$H(s) = \frac{N_H(s)}{D_H(s)}$$



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + k\,G(s)H(s)} = \frac{N_G/D_G}{1 + k\,\frac{N_G}{D_G}\cdot\frac{N_H}{D_H}} = \frac{N_G D_H}{D_G D_H + k\,N_G N_H}$$

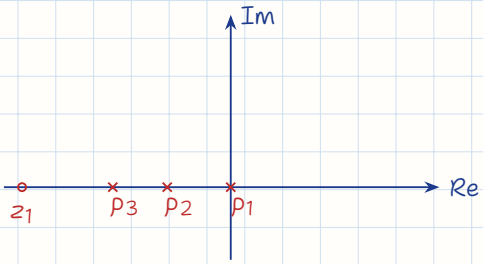
CLTF $\begin{cases} \text{zeros of TF} \Rightarrow \text{"Zeroes" of } G + \text{"poles" of } H \\ \text{poles of T.F} \Rightarrow D_G D_H + k N_G N_H = 0 \longrightarrow k = 0 \Rightarrow D_G D_H = 0 \quad (D_G \rightarrow \text{poles of } G,\ D_H \rightarrow \text{poles of } H) \end{cases}$

By changing 'k' roots of the characteristic equation change from 'poles' to 'zero' of the 'open loop' T.F.

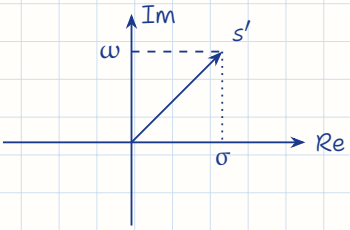
$$k \uparrow \infty \Rightarrow N_G N_H = -\frac{D_G D_H}{k} \to 0 \quad (N_G \rightarrow \text{zeroes of } G,\ N_H \rightarrow \text{zeroes of } H)$$

orig. p. 2

$$G(s) = \frac{s+5}{s(s+1)(s+2)} \qquad \begin{array}{l} \rightarrow z_1 = -5 \\ \rightarrow p_1 = 0,\ p_2 = -1,\ p_3 = -2 \end{array}$$



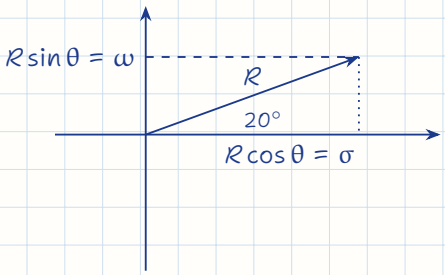
Vector representation in complex plane



$$\sigma + j\omega = \mathcal{R} e^{j\theta} = \mathcal{R}\angle\theta$$
$$\mathcal{R} = \sqrt{\sigma^2 + \omega^2}$$
$$\theta = \tan^{-1}\left(\frac{\omega}{\sigma}\right)$$

origin of complex plane to s'.

$$\begin{cases} \mathcal{R} = 10 \\ \theta = 20 \end{cases}$$

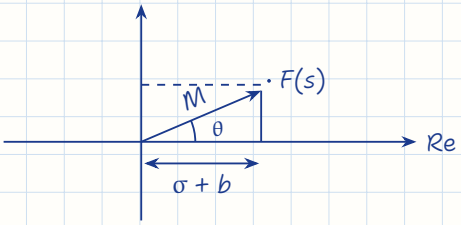


$$s = \mathcal{R}\cos\theta + j\mathcal{R}\sin\theta$$

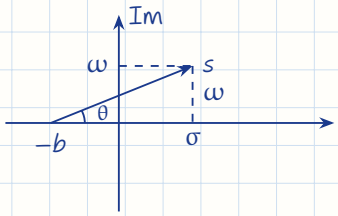
s is complex

$F(s) = s + b$ for any complex s, we'll end up to as a complex number

$$s = \sigma + j\omega$$
$$F(s) = (\sigma + b) + j\omega = M\angle\theta$$
$$M = \sqrt{(\sigma + b)^2 + \omega^2}$$
$$\theta = \tan^{-1}\left(\frac{\omega}{\sigma + b}\right)$$



Starting from origin of s plane



Starting from origin of F(s)

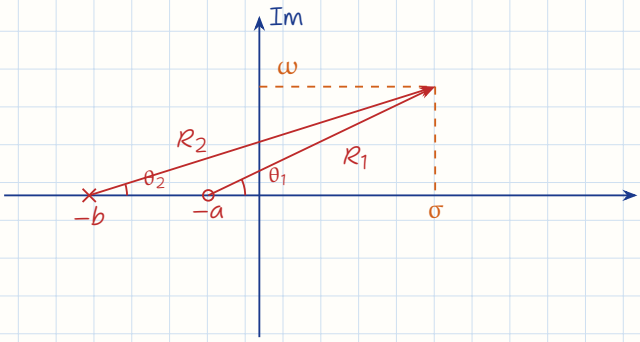
orig. p. 3

$$F(s) = \frac{s+a}{s+b} \qquad \text{origin of } s+a \text{ (numerator)}$$

$$s = \sigma + j\omega \quad \Rightarrow \quad F(s) = \frac{\sigma + j\omega + a}{\sigma + j\omega + b} \times \frac{(\sigma + b) - j\omega}{(\sigma + b) - j\omega}$$

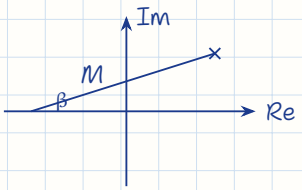
$$F(s) = \frac{\overbrace{[(\sigma+b)^2 + j\omega]}^{R_1 e^{j\theta_1}} \overbrace{[(\sigma+b) - j\omega]}^{R_2 e^{j\theta_2}}}{(\sigma+b)^2 + \omega^2} \; [1]$$

$$= \frac{R_1}{\alpha} e^{j\theta_1} + \frac{R_2}{\alpha} e^{j\theta_2} \; [2]$$



$$F(s) = \frac{M_1 e^{j\beta_1}}{M_2 e^{j\beta_2}} = \left|\frac{M_1}{M_2}\right| e^{j(\beta_1 - \beta_2)} = M e^{j\beta}$$

$$F(s) = \frac{\prod_{i=1}^{N}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} = \frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$



at any s, the function value is a complex number (in general) which has a "magnitude" and a "phase"

orig. p. 4

$$M = \frac{M_1 \cdot M_2 \cdots M_N}{M_1' M_2' \cdots M_m'}$$

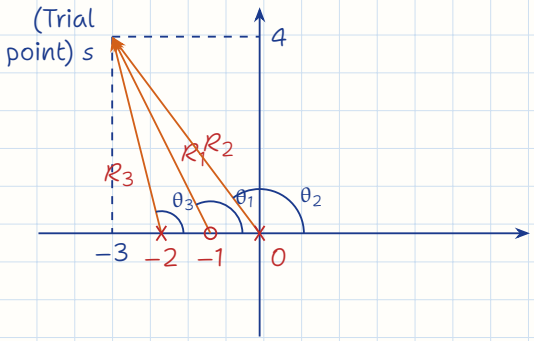
$$\theta = \sum \text{Phases of Numerator} - \sum \text{phases of denominator}$$

$$F(s) = \frac{s+1}{(s+2)s}$$

What is the function value at $s = -3 + j4$

[1] Claude's note: probably $(\sigma + a) + j\omega$, i.e. the numerator $\sigma + j\omega + a$.

[2] Claude's note: the product of the two terms would give $\frac{R_1 R_2}{\alpha} e^{j(\theta_1+\theta_2)}$, not a sum.



$$\text{For } \vec{R_1} \begin{cases} M_1 \text{ magnitude} = \sqrt{2^2 + 4^2} \\ \text{phase } \beta_1 = \pi - \tan^{-1}\left(\frac{4}{2}\right) \end{cases}$$

$$\text{For } \vec{R_2} \begin{cases} M_2 \text{ mag} = \sqrt{3^2 + 4^2} \\ \beta_2 = \pi - \tan^{-1}\left(\frac{4}{3}\right) \end{cases}$$

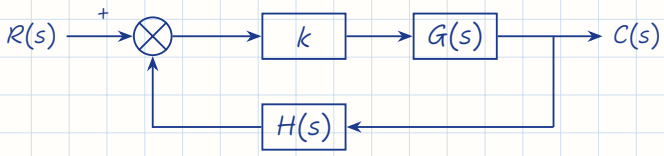
$$\text{For } \vec{R_3} \begin{cases} M_3 = \sqrt{1^2 + 4^2} \\ \beta_3 = \pi - \tan^{-1}(4/1) \end{cases}$$

$$M = \frac{M_1}{M_2 M_3}$$

$$\theta = \underbrace{\beta_1}_{\text{numerator}} - \underbrace{\beta_2 - \beta_3}_{\text{Denominator}}$$

orig. p. 5

4 We need some criteria to check if the trial points (s) on the complex plane can be accepted as root of the characteristic equation



$$\frac{C(s)}{R(s)} = \frac{k\,G(s)}{1 + k\,G(s)H(s)}$$

$$1 + k\,G(s)H(s) = 0 \;\Rightarrow\; G(s)H(s) = -\frac{1}{k} + 0j$$

magnitude: $|G(s)H(s)| = \left|\frac{1}{k}\right|$

phase: $\angle G(s)H(s) = \tan^{-1}\left(\frac{0}{-1/k}\right) = \pm 180(2i+1) \qquad i = 0,1,\ldots$

↑ because its negative

Any point like 's' which satisfies both conditions would be our point of interest (root of characteristic eq)

The locus of points satisfying the phase condition constitutes the root locus plot of the system.

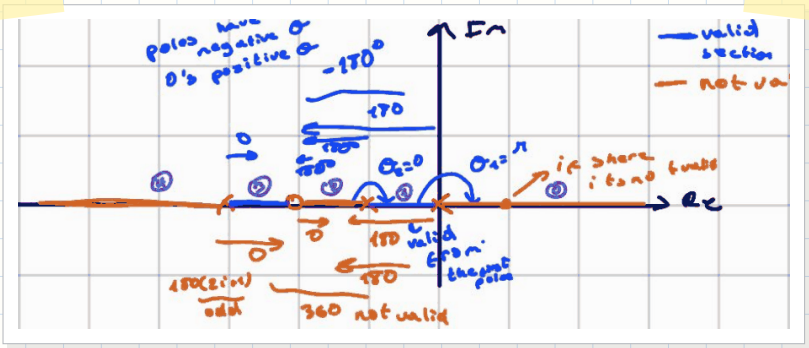
Steps of Drawing

1- Find the OL zeros and poles and show on the s plane

$$1 + k\,G(s)H(s) \Rightarrow 1 + k\,\frac{Q(s)}{P(s)} = 0 \Rightarrow P(s) + k\,Q(s) = 0$$

$$\begin{cases} \text{if } k = 0 \Rightarrow P(s) = 0 \Rightarrow \text{poles of OLTF} & \text{x symbol} \\ \text{if } k \uparrow \infty \Rightarrow Q(s) = -\dfrac{P(s)}{k} \to 0 \Rightarrow Q(s) = 0 \Rightarrow \text{zeros of OLTF} & \text{o symbol} \end{cases}$$

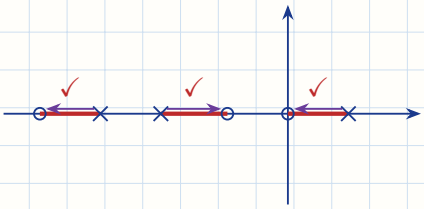
2- Check the validity of point on 'Re' axis



original drawing, p. 5

orig. p. 6

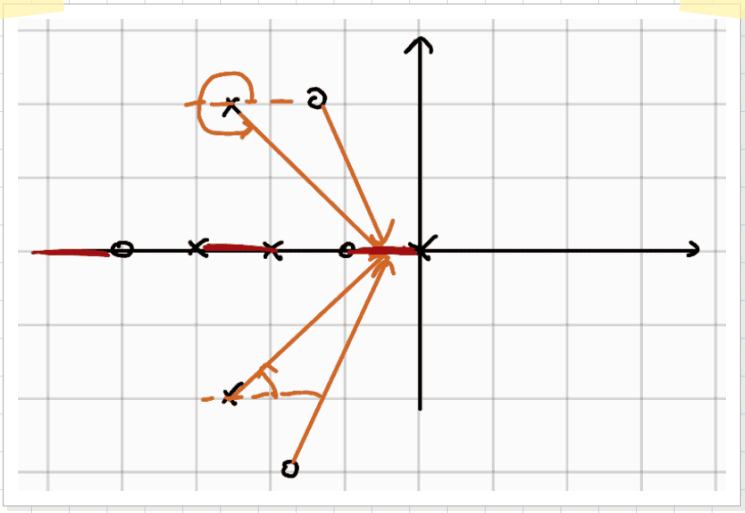
⋆ So if the number of zeros/ poles on the right hand side of the zone is 'odd', then it passes the phase condition



if $k = 0 \longrightarrow k = \infty$

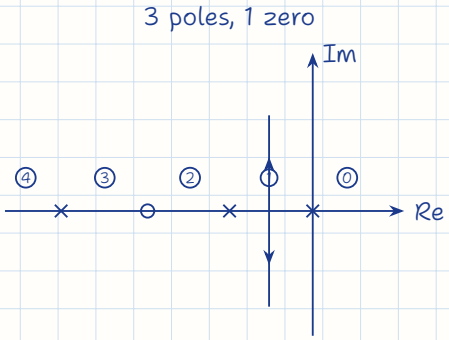
→ roots move

⋆ poles of CL
Start at poles of O.L.
end at zero " O.L.



original drawing, p. 6

Complex poles/zeroes not effect the acceptable zone at the real axis



All poles have to married with zeros :)
if they can't they go to infinity :(

3- Asymptote determine

if the number of poles and zeros are not the same

Then there will be some asymptotes for the root locus

poles : n $\Rightarrow n-m$: number of asymptotes

zeros : m

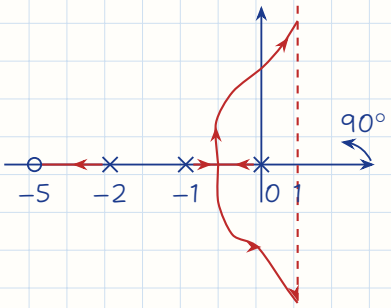
'n' is usually larger than 'm'

⋆ Number of Branches is equal to 'n'

orig. p. 7

Angle of asymptotes $\Phi = \dfrac{(2i+1)180}{n-m}, \quad i = 0,1,2,\ldots,n-m-1$

⋆ Centroide of asymptotes $s' = \dfrac{\sum \text{real part of poles} - \sum \text{real part of zeros}}{n-m}$



$n-m = 2$ = No of asymp

$$\Phi = \frac{(2i+1)180}{2} = (2i+1)90$$

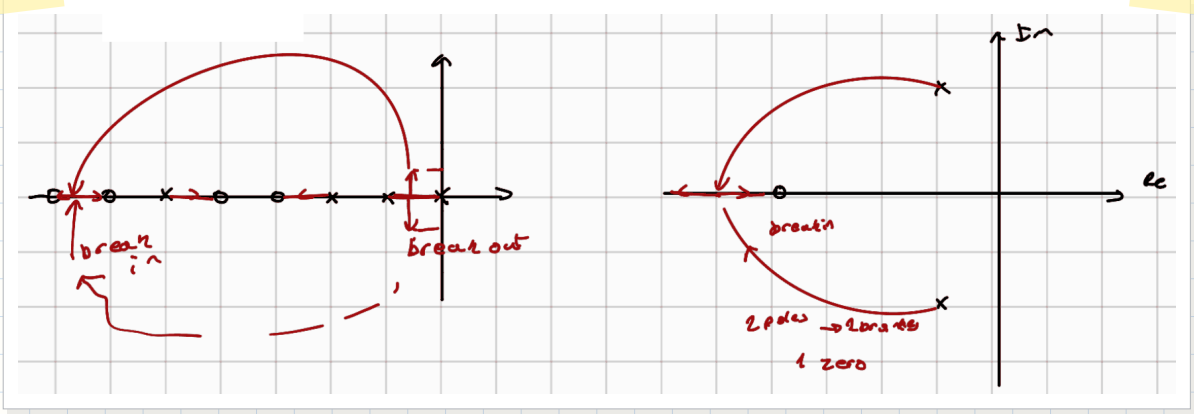
$i = 0 \rightarrow \Phi_1 = 90$

$i = 1 \rightarrow \Phi_2 = 270$

centroid: $\dfrac{(0-1-2)-(-5)}{2} = +1$

Determine the breaking points

⋆ Branches leave (break out) or enter (break in) the real axis at 90°



original drawing, p. 7

The characteristic eq

$$1 + k\frac{Q(s)}{P(s)} = 0$$

$$P(s) + k\,Q(s) = 0$$

$$\Rightarrow k = -\frac{P(s)}{Q(s)}$$

repeated root occurs at point of break in/out $\Rightarrow \frac{dk}{ds} = 0$

$$\hookrightarrow P'Q - Q'P = 0$$

orig. p. 8

All break points are solution of this polynomial. But not all solutions are break in/out points

Only root of the given expression are acceptable that give a positive gain!

The other approach is that the break in/out point must be on the valid zones of the real axis.