Kontrol Sistemleri

Root Locus (devam)

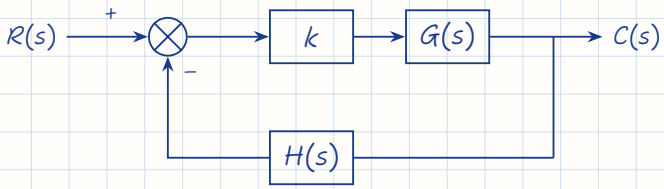
Original notes: Control-Systems-lecture-11.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Root locus

orig. p. 1



$$\frac{C(s)}{R(s)} = \frac{k\,G(s)}{1 + k\,G(s)H(s)}$$

$$1 + k\,G(s)H(s) = 0 \;\Rightarrow\; G(s)H(s) = -\frac{1}{k}$$

↗ mag: $|G(s)H(s)| = \frac{1}{k}$

↘ phase: $\angle G(s)H(s) = \pm 180(2i+1), \quad i = 0,1,\ldots$

$$\begin{cases} k = 0 \Rightarrow \text{poles of the OLTF} \\ k \uparrow \infty \Rightarrow \text{zero } " \; " \; " \end{cases}$$

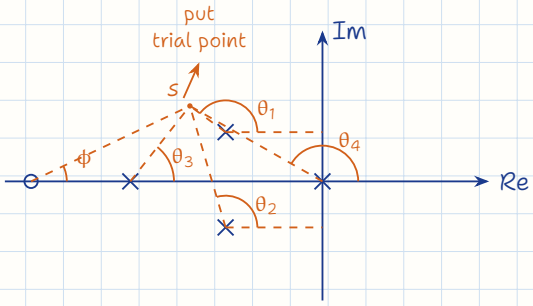
n: number of poles
m: " of zeros

$$\begin{cases} n: \text{ branches} \\ n - m: \text{ asymptotes} \end{cases}$$

$$\Phi = \frac{\pm 180\,(2i+1)}{n - m}$$

$$\text{Centroid: } \frac{\sum \text{Real poles} - \sum \text{Real zeros}}{n - m}$$

5 - Complex conjugate poles



$\angle$trial point $= \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 = 180$

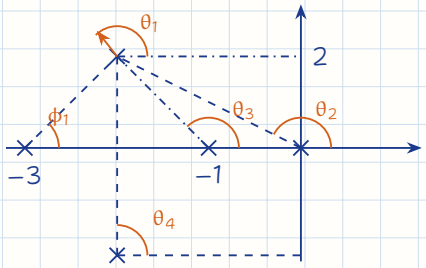
$\hookrightarrow \theta_1 = 180 - \phi_1 + \theta_2 + \theta_3 + \theta_4$

$\downarrow$ departure angle from C.C pole

$\theta_i = 180 - \sum$ angles from the complex conjugate pole and other zero $+ \sum$ angles from ?*ditto marks* pole

Departure angle from poles?

arrival angle " zeros



orig. p. 2

$\theta_1 = 180 - \phi_1 + \theta_2 + \theta_3 + \theta_4$

$= 180 - \tan^{-1}\left(\frac{2}{1}\right) + \tan^{-1}\left(\frac{2}{-2}\right)$

$+ \tan^{-1}\left(\frac{2}{-1}\right) + 90°$

(arrow from the underlined term:) $\pi - \tan^{-1}\left(\frac{2}{1}\right)$



original drawing, p. 2

$\phi_1 + \phi_2 + \phi_3 - \theta_1 - \theta_2 - \theta_3 = 180$

$\phi_1 = 180 - \phi_2 - \phi_3 + \theta_1 + \theta_2 + \theta_3$

$= 180 - \sum$ angles from the trial point (cc zero) to other zeros

$+ \sum$ " " " — poles

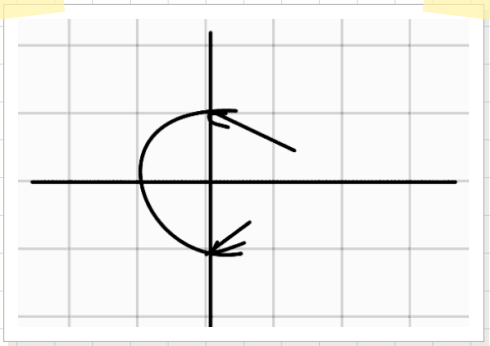
Step 6 Intersection with the $j\omega$ axis

I) $s = \sigma^0 + j\omega$ substitute $s = j\omega$ in the characteristic

$$s^2 + 2s + 6 = 0$$
$$s^2 + 2s + k = 0$$
$$(j\omega)^2 + 2j\omega + k = 0$$
$$-\omega^2 + 2j\omega + k = 0$$

$$\Rightarrow \quad \begin{matrix} \text{Im} \\ \text{Re} \end{matrix} \begin{cases} 2\omega = 0 \\ k - \omega^2 = 0 \end{cases} \Rightarrow \begin{cases} \omega = 0 \\ k = \omega^2 = 0 \end{cases}$$

$$s^3 + 2s^2 + 3s + 6 + k = 0$$



original drawing, p. 2

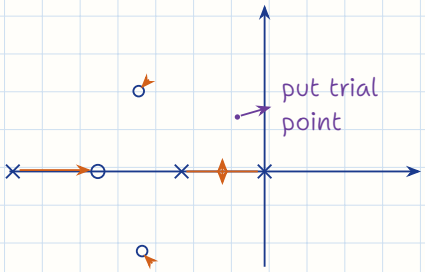
orig. p. 3

II) characteristic eq

Routh table

if one of the components of the first column is zero

⋆ make one full row zero!



satisfy the phase
190 ↶
170 ↓

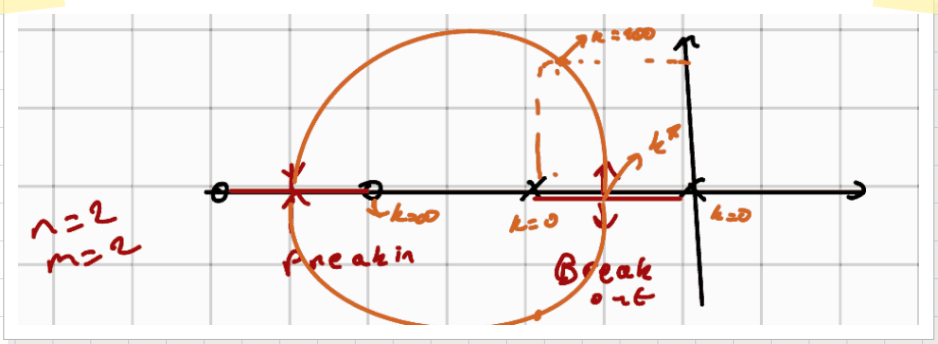
or $s = \sigma + j\omega$

So Step 7- Find the location of root locus of the complex poles!

Method I) do numerical trial & error to discover the locations of 's' which satisfy the phase condition

Method II) Substitute s as $s = \sigma + j\omega$ in the characteristic equation and mathematically find the phase condition.

$$1 + k\,G(s)H(s) = 0$$
$$G(s)H(s) = -\frac{1}{k}$$
$$\hookrightarrow \text{mag} = |G(s)H(s)| = \frac{1}{k}$$
$$\hookrightarrow \text{phase: } \angle G(s)H(s) = 180(2i + 1)$$

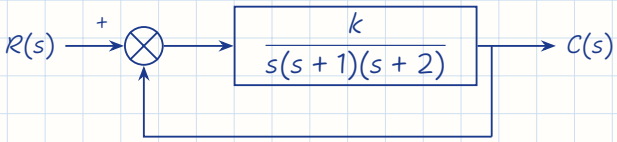


original drawing, p. 3

orig. p. 4

$$|G(s)H(s)| = \frac{\prod_{i=1}^{m} l_i}{\prod_{j=1}^{n} l_j}$$

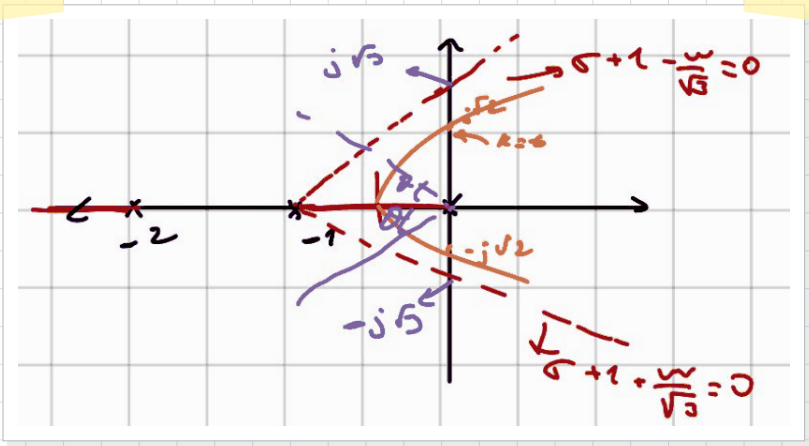
l_i, l_j: magnitudes of individual complex function in numerator and denominator



-plot the root locus
-tune k so that $\zeta = 0.5$ for the dominant poles

$\angle G(s)H(s) = 180(2i+1)$
$-\angle s - \angle s{+}1 - \angle s{+}2$ = total phase

$\begin{cases} n = 3 \\ m = 0 \end{cases}$ $\begin{cases} 3 \text{ branches} \\ 3 \text{ asymptotes} \end{cases}$



original drawing, p. 4

$$\text{angles of asymptotes} = \frac{\pm 180(2i+1)}{n-m} = \pm 60(2i+1) \quad \begin{matrix} \to 60 \\ \to -60 \\ \searrow 180 \end{matrix}$$

$$\text{Centroid: } \frac{\sum \text{real part of poles} - \sum \text{real parts of zeros}}{n-m} = \frac{0-1-2-0}{3} = -1$$

$s \nearrow \infty$ $\quad s \approx s+1 \approx s+2$

$$G(s) = \frac{k}{s(s+1)(s+2)} = \frac{k}{s^3+3s^2+2s} \approx \frac{k}{s^3+3s^2+3s+1}$$

for large s values

Approximate $\bar{G}(s) \approx \dfrac{k}{(s+1)^3}$

$$\angle\bar{G}(s) = -3\angle s+1 = \pm 180(2i+1)$$
$$\Rightarrow \underline{\angle(s+1)} = -60(2i+1)$$

$$s = \sigma + j\omega \Rightarrow \angle(\sigma + j\omega + 1) \approx \tan^{-1}\left(\frac{\omega}{\sigma+1}\right) = \pm 60(2i+1) \quad \begin{matrix} \rightarrow -60 \\ \rightarrow 60 \\ \searrow 180 \end{matrix}$$

$$\tan^{-1}\left(\frac{\omega}{\sigma+1}\right) = \pm 60, 180$$

orig. p. 5

tan ↓

$$\frac{\omega}{\sigma+1} = \begin{matrix} \rightarrow \tan(60) = \sqrt{3} \\ \rightarrow \tan(-60) = -\sqrt{3} \\ \rightarrow \tan(180) = 0 \end{matrix}$$

$$\Rightarrow \begin{cases} \omega = \sqrt{3}\,(\sigma+1) \\ \omega = -\sqrt{3}\,(\sigma+1) \\ \omega = 0 \end{cases} \quad \text{asymptote equation}$$

The characteristic equ.

$$1 + \frac{k}{s(s+1)(s+2)} = 0$$

$$\Rightarrow \frac{s(s+1)(s+2)+k}{s(s+1)(s+2)} = 0 \Rightarrow k = -s(s+1)(s+2)$$

$$\frac{dk}{ds} = 0 \Rightarrow 3s^2 + 6s + 2 = 0 \Rightarrow \begin{cases} s_1 = -0{,}42 \quad \checkmark \text{ is acceptable between 0 and -1} \\ s_2 = -1{,}57 \quad ?\,\times \end{cases}$$

$$\begin{cases} k(s_1) > 0 \quad \checkmark \\ k(s_2) < 0 \quad \times \end{cases}$$

Intersection with $j\omega$ axis

$s^3 + 3s^2 + 2s + k = 0 \qquad k > 0$

s^3	1	2
s^2	3	k
s	$\frac{6-k}{3}$	0
s^0	k	

$6 - k = 0 \Rightarrow k = 0$[a]

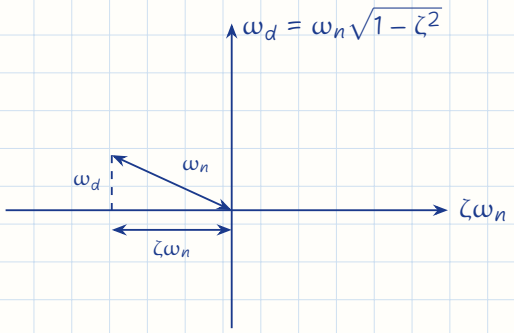
Auxilary equation
$3s^2 + k = 0$
Roots of this equation are some of the roots for the characteristic equation
$3s^2 + 6 = 0 \Rightarrow s^2 = -2 \Rightarrow s_{1,2} = \pm j\sqrt{2}$

[a] Claude's note: probably $k = 6$.

$k < 6 \qquad 0 \le k \le 6$

orig. p. 6

$\zeta = 0{,}5 \qquad \theta = \cos^{-1}\zeta = \cos^{-1} 0{,}5 \qquad \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta = ?$



$$\cos\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \qquad \theta = \cos^{-1}\zeta$$

$$\frac{k}{s(s+1)(s+2)}$$

$$s = \sigma + j\omega$$

$$G(s) = -\angle s - \angle s{+}1 - \angle s{+}2 = 180(2i+1)$$

$$= -\angle\sigma{+}j\omega - \angle\sigma{+}1{+}j\omega - \angle\sigma{+}2{+}j\omega =$$

$$= -\tan^{-1}\left(\frac{\omega}{\sigma}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+1}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+2}\right) = 180(2i+1)$$

$$\Rightarrow \underline{-\tan^{-1}\left(\frac{\omega}{\sigma}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+1}\right)} = \underline{\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) + 180(2i+1)}$$

★ $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$$= \frac{\tan\left(\tan^{-1}\frac{\omega}{\sigma}\right) + \tan\left(\tan^{-1}\frac{\omega}{\sigma+1}\right)}{1 - \tan\left(\tan^{-1}\frac{\omega}{\sigma}\right)\tan\left(\tan^{-1}\frac{\omega}{\sigma+1}\right)}$$

$$= \frac{\tan\left(\tan^{-1}\frac{\omega}{\sigma+2}\right) + 0}{1 - \tan\left(\tan^{-1}\frac{\omega}{\sigma+2}\right) \times 0}$$

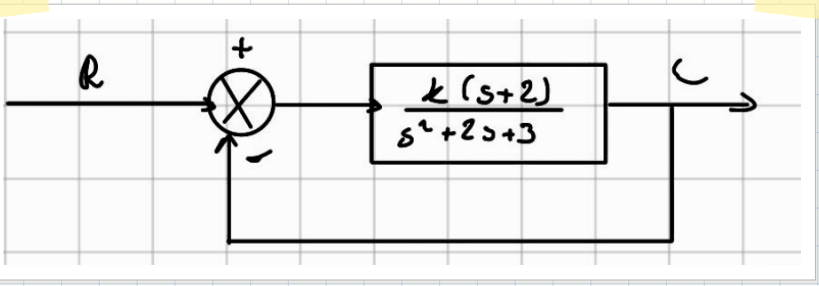
$$\Rightarrow \left(\frac{\frac{\omega}{\sigma} + \frac{\omega}{\sigma+1}}{\underbrace{1 - \left(\frac{\omega}{\sigma}\right)\left(\frac{\omega}{\sigma+1}\right)}_{\beta}} = \frac{\omega}{\sigma+2} \right) \beta$$

orig. p. 7

$$\frac{k\,G(s)}{1 + k\,G(s)H(s) = 0}$$

$$|G(s)H(s)| = \frac{1}{k}$$

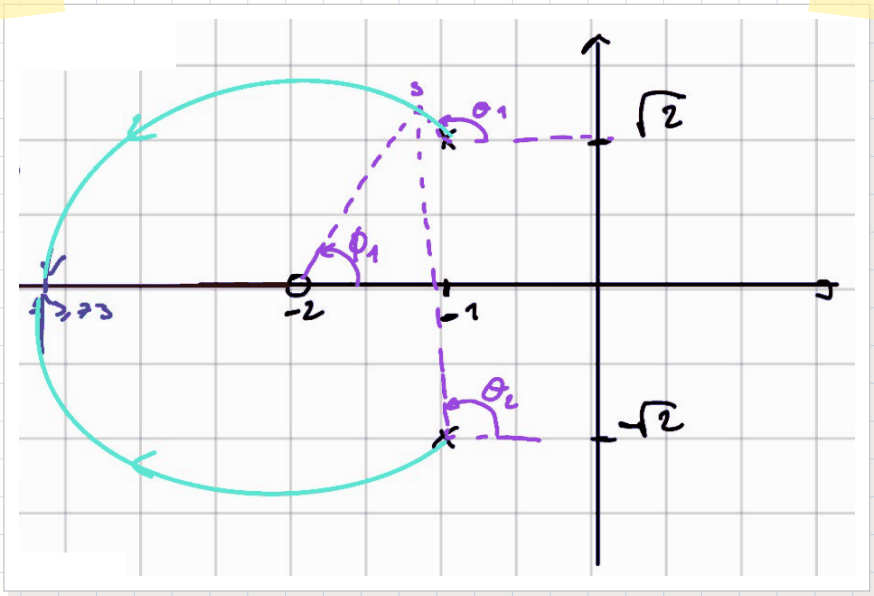
checking the magnitude condition at ? $s = s_1$



original drawing, p. 7

$$G(s) = \text{OLTF} = \frac{k(s+2)}{s^2+2s+3}$$

$$z_1 = -2 \qquad p_{1,2} = -1 \mp j\sqrt{2}$$



original drawing, p. 7

n = number of poles = 2

m = " " zeroes = 1

$$\text{centroid} = \frac{\sum \text{real of poles} - \sum \text{real of zero}}{1}$$

$$= \frac{-1-1-(-2)}{1} = 0$$

break in point:

$$1 + \frac{k(s+2)}{s^2+2s+3} = 0$$

$$k = -\frac{s^2+2s+3}{s+2}$$

$$\frac{dk}{ds} = 0 \Rightarrow \checkmark \begin{cases} s_1 = -0{,}268 & \times \\ s_2 = -3{,}732 & \checkmark \text{ acceptable} \end{cases}$$

$$\phi_1 - \theta_1 - \theta_2 = 180$$

$$-\theta_1 = 180 - \phi_1 + \theta_2$$

orig. p. 8

Departure angles from complex poles

$$\theta_1 = 180 - 90 + 55 = 145^\circ$$

for the complex conjugate pair the department angle is 145°

$$\angle(s+2) - \angle(s^2+2s+3) = 180(2i+1)$$

$$s = \sigma + j\omega$$

$$\angle(s+2) - \angle(s+1-j\sqrt{2}) - \angle(s+1+j\sqrt{2}) = 180(2i+1)$$

$$\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) - \tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right) - \tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) = 180(2i+1)$$

$$\tan\left[\tan^{-1}\frac{\omega}{\sigma+2} - \tan^{-1}\frac{\omega-\sqrt{2}}{\sigma+1} = \tan^{-1}\frac{\omega+\sqrt{2}}{\sigma+1} + 180(2i+1)\right]$$

$$\frac{2\omega(\sigma+1)}{(\sigma+1)^2 - (\omega^2-2)} = \frac{\omega}{\sigma+2} \Rightarrow \omega\left[(\sigma+2)^2 + \omega^2 - 3\right] = 0$$

if $\omega = 0$

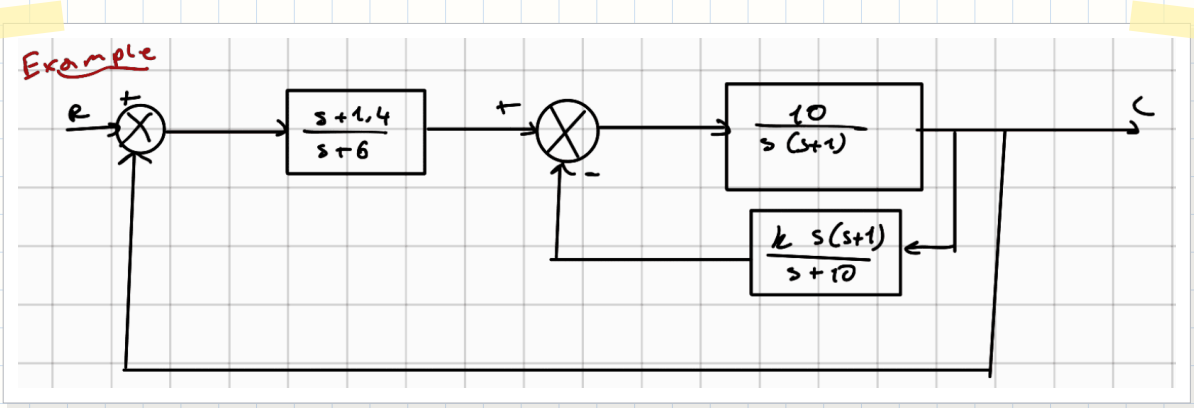
if $(\sigma+2)^2 + \omega^2 = (\sqrt{3})^2$

$(-2,0)$ radius $= \sqrt{3}$

and $(x-x_0)^2 + (y-y_0)^2 = R^2$

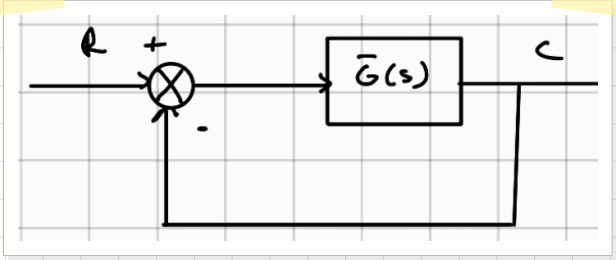
circle

(x_0, y_0), radius R



original drawing, p. 8

$$\frac{k\,G(s)}{1 + k\,G(s)H(s)}$$



original drawing, p. 8

$$\bar{G}(s) = \left(\frac{s+1{,}4}{s+6}\right)\left(\frac{10(s+10)}{s(s+1)(s+10+10k)}\right)$$

orig. p. 9

Characteristic eq

$$1 + \bar{G}(s) = 0$$

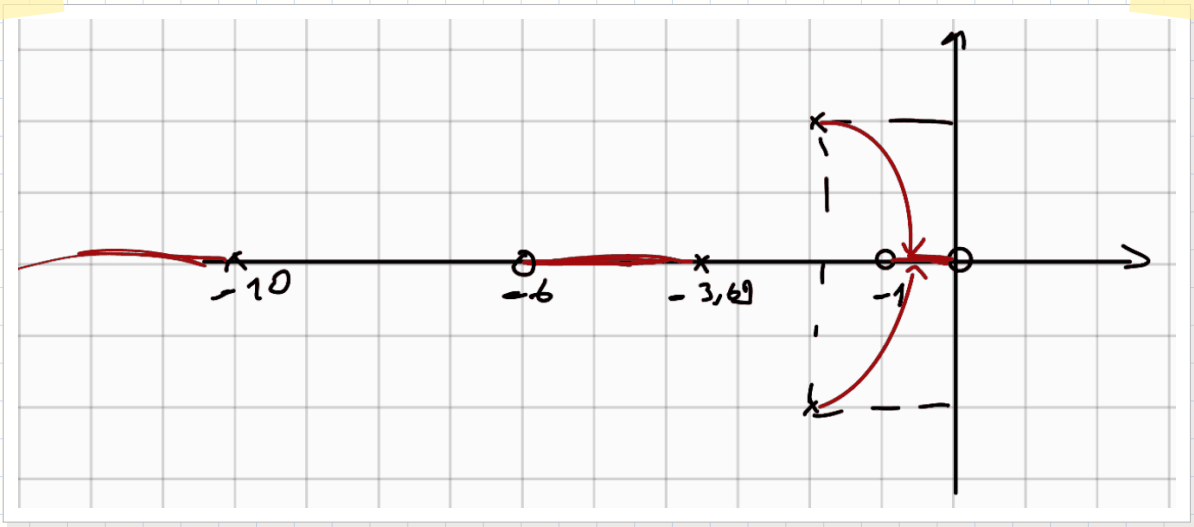
$$\frac{(s+6)\,s(s+1)(s+10+10k) + (s+1{,}4)\,10\,(s+10)}{\beta} = 0$$

$$\Rightarrow \underbrace{s(s+1)(s+6)(s+10)}_{\alpha_1} + 10k\,s(s+1)(s+6) + \underbrace{10(s+1{,}4)(s+10)}_{\alpha_2}\quad (s+10) = 0$$

$$1 + k\,\frac{10\,s(s+1)(s+6)}{s(s+1)(s+6)(s+10) + 10(s+1{,}4)(s+10)} = 0$$

$$z_1 = 0 \quad z_2 = -1 \quad z_3 = -6$$

$$p_1 = -10, \quad p_2 = -3{,}69, \quad p_{3,4} = -1{,}65 \pm \text{?}\,1{,}02\,j$$



original drawing, p. 9