Kontrol Sistemleri

Modeling, Laplace Transform & Partial Fractions

Original notes: Control-Systems-lecture-2.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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orig. p. 1

Modeling Dynamical Systems

→ Expression of system's behavior by some mathematical relations obtained by setting reasonable assumptions. Differential Eq, Integral Eq, Integro diff Eq

Differential Equations

- partial: number of independent variables is more than one — continuous system

$$a_1\frac{\partial^2 c}{\partial x^2} + a_2\frac{\partial^2 c}{\partial y^2} + a_3\frac{\partial^2 c}{\partial x\,\partial y} + a_4\frac{\partial^2 c}{\partial t^2} = r(t)$$ (Linear PDE, order=2, degree=1)

- ordinary: — Lumped/discretized

$$a_1\frac{d^2c}{dt^2} + a_2\frac{dc}{dt} + a_3 c = r(t)$$ (Linear ODE, order=2)

ODE, 2nd order, 3rd degree ⟷

$$a_1\frac{d^2c}{dt^2} + a_2\frac{dc}{dt} + a_3c^3 + a_4c^2 = r(t)$$ → non linear ×

* Lumped, discretizing (ayrık)

System Dynamics:
Response due to Initial conditions.

Dynamical system
→ A system changing to the time

Systems
- → static ×
- non-static
  - → Dynamic (deterministic) ✓
  - → stochastic ×

We have to talk about linear ODE's that change respect time

We use laplace to change differential equation to algebraic equation. ($s^2 + s + a = r$ algebraic)

orig. p. 2

Laplace Transform

$$\underbrace{\mathcal{L}}_{\text{laplace operator}}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Differential theorem in laplace transform

$$\mathcal{L}\left(\frac{d^n f(t)}{dt^n}\right) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

output is y input is x

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_m x$$

(the order is n) $(n > m)$

Most general ODE for a system.

Assuming that initial conditions are zero (to ignore transient respons)

$$a_1 \ddot{x} + a_2 \dot{x} + a_3 \overbrace{x}^{\text{output}} = \underbrace{r(t)}_{\text{input}}$$

$$x(t=0) = x_0$$
$$\dot{x}(t=0) = \dot{x}_0$$

Initial condition: n-1'e kadar $y^{n-1}(0)$, $y^{(n-2)}(0)$, ..., $y(0)$
n tane bulunmakta

Laplace transform ⇓

$$(a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n)\,Y(s) = (b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_m)\,X(s)$$

$$\text{Transfer function} = \frac{\text{output}}{\text{Input}} = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} = G(s)$$

* The max order of derivative define the output of the system
this is independent to input (intrinsic definition of system)

$$G(s) = \frac{Y(s)}{X(s)} \rightarrow \underbrace{Y(s)}_{\text{response}} = \underbrace{X(s)}_{\text{input}}\,G(s)$$

$$\mathcal{L}^{-1} \rightarrow y(t) = \int_0^t x(\tau)\,g(t-\tau)\,d\tau = \int_0^t x(t-\tau)\,g(\tau)\,d\tau$$

orig. p. 3

Impulse function in frequency domain

$$X(s) = 1 \quad \rightarrow \quad Y(s) = G(s)$$

impulse response = T.F. (Transfer function)

$$\mathcal{L}^{-1}[G(s)] = g(t)$$

zero initial condition

$$\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$$

$$\xrightarrow{\mathcal{L}} s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

$$\hookrightarrow \frac{C(s)}{R(s)} = \underbrace{\frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}}_{\text{Dynamical system representation}} = G(s) \quad \text{T.F.}$$

$$\Rightarrow C(s) = R(s)\,G(s)$$

Find the D.E for the transfer function

$$G(s) = \frac{2s + 1}{s^2 + 6s + 2} = \frac{C(s)}{R(s)} \begin{matrix}\text{output}\\ \text{input}\end{matrix}$$

$$(2s + 1)R(s) = C(s)(s^2 + 6s + 2)$$

$$\mathcal{L}^{-1} \rightarrow 2\dot{r}(t) + r(t) = \frac{d^2c(t)}{dt^2} + 6\frac{dc(t)}{dt} + 2c(t)$$

Linearity theorem:

$$\mathcal{L}[kf(t)] = K \cdot F(s)$$

$$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s) \quad \text{Superposition.}$$

Freq shift $\mathcal{L}[e^{-at}f(t)] = F(s + a)$

Time shift $\mathcal{L}[f(t - T)] = e^{-sT}F(s)$

Scaling:

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

* Differention:

$$\mathcal{L}\left(\frac{d^nf(t)}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

orig. p. 4

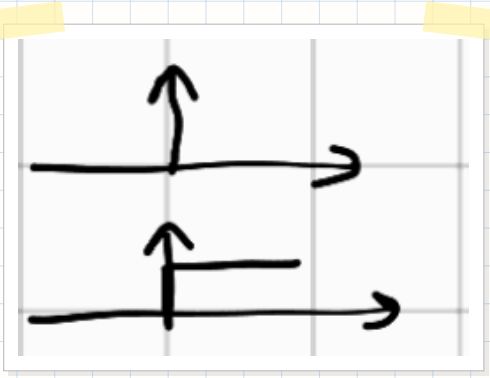
* Integration:

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}$$

* Final value: $f(\infty) = \lim_{s\to 0} sF(s)$

Initial value $f(0) = \lim_{s\to\infty} sF(s)$

Table to know



original drawing, p. 4

$f(t)$	$F(s)$
impulse $\delta(t)$	1
step $u(t)$	$1/s$
Ramp t	$1/s^2$
$\frac{t^{n-1}}{(n-1)!}$	$1/s^n$
e^{-at}	$\frac{1}{s+a}$
$\frac{e^{-at}\,t^{n-1}}{(n-1)!}$	$\frac{1}{(s+a)^n}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$

Partial fraction expansion (PFE)

T.F. $G(s) = \frac{B(s)}{A(s)}$ $\rightarrow$ numerator $\rightarrow$ roots of numerator polynomial = zero
$\rightarrow$ denominator $\rightarrow$ " " denominator " poles

case 1: Poles are distinct and with multiplicity of unity!
tekrarlanmaz, the poles are unique, differents, not repeat

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n}$$

poles are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.
zeros m

orig. p. 5

$$= \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \cdots + \frac{r_n}{s-\lambda_n}$$

r_n = residues of the function

How to obtain the redisues.

$n > m$: strictly proper (for all physical systems) ✓

$n = m$: proper

$n < m$: not proper

Example:

$$\frac{s^2-2}{s^3+3s^2+2s} = \frac{s^2-2}{s(s^2+3s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

Roots are $\begin{cases} s_1 = 0 \\ s_2 = -1 \\ s_3 = -2 \end{cases}$

$$= \frac{r_1(s+1)(s+2) + r_2(s)(s+2) + r_3(s)(s+1)}{s(s+1)(s+2)}$$

$$r_1(s^2+2s+s+2) + r_2(s^2+2s) + r_3(s^2+s) = s^2-2$$

$$\begin{cases} s^0: & 2r_1 = -2 \\ s^1: & 3r_1 + 2r_2 + r_3 = 0 \\ s^2: & r_1 + r_2 + r_3 = 1 \end{cases}$$

* Second method of residue calculation: $r_k = (s-\lambda_k)F(s)\Big|_{s=\lambda_k}$

$$r_1 = sF(s)\Big|_{s=0}$$

$$= \frac{s^2-2}{s^2+3s+2}\Big|_{s=0} = \frac{-2}{2} = -1 \checkmark$$

orig. p. 6

$$G(s) = \frac{B(s)}{A(s)} \rightarrow \text{roots are } \lambda_k$$

$$r_k = \frac{(s-\lambda_k)B(s)}{A(s)}\Big|_{s=\lambda_k} \quad * \text{ L'Hopital}$$

$$\lim_{s\to\lambda_k} \frac{\text{derivative of num.}}{\text{derivative of den.}} = \lim_{s\to\lambda_k} \frac{B(s) + (s-\lambda_k)B'(s)}{A'(s)}$$

more efficient

$$r_k = \frac{B(s)}{A'(s)}\Big|_{s=\lambda_k}$$ ← **önemli**

Example

$$V^{(3)} - V = 0 \qquad \begin{matrix} V(0) = 1 \\ \dot{V}(0) = 0 \\ \ddot{V}(0) = 0 \end{matrix}$$

$$\frac{d^3V}{dt^3} = V^{(3)}$$

$$\mathcal{L} \rightarrow s^3V(s) - s^2V(0) - s\dot{V}(0) - \ddot{V}(0) - V(s) = 0$$

$$\hookrightarrow V(s)(s^3 - 1) = s^2 \Rightarrow V(s) = \frac{s^2}{s^3 - 1}$$

$$\begin{array}{r|l} s^3 - 1 & s - 1 \\ \underline{-s^3 + s^2} & s^2 + s + 1 \\ s^2 - 1 & \\ \underline{-s^2 + s} & \\ s - 1 & \end{array}$$

$$s^3 - 1 = (s - 1)(s^2 + s + 1)$$

$$\downarrow s_1 = 1 \qquad \searrow s_{2,3} = \frac{-1 \pm j\sqrt{3}}{2}$$

three poles of multiplicity one

$$V(s) = \frac{s^2}{s^3 - 1} = \frac{r_1}{s - 1} + \frac{r_2}{s + \frac{1}{2} + \frac{\sqrt{3}}{2}j} + \frac{r_3}{s + \frac{1}{2} - \frac{\sqrt{3}}{2}j}$$

orig. p. 7

$$r_k = \left.\frac{B(s)}{A'(s)}\right|_{s=\lambda_k}$$

$$B(s) = s^2$$

$$A(s) = s^3 - 1 \Rightarrow A'(s) = 3s^2$$

$$r_k = \left.\frac{s^2}{3s^2}\right|_{s=\lambda_k}$$

$$r_1 = \frac{(1)^2}{3(1)^2} = \frac{1}{3}$$

$$r_2 = \frac{1}{3}$$

$$r_3 = \frac{1}{3}$$

$$V(s) = \frac{s^2}{s^3 - 1} = \frac{1/3}{s - 1} + \frac{1/3}{s + \frac{1}{2} + \frac{\sqrt{3}}{2}j} + \frac{1/3}{s + \frac{1}{2} - \frac{\sqrt{3}}{2}j}$$

$$V(t) = \mathcal{L}^{-1}\{V(s)\} = \frac{1}{3}\left[e^{+t} + e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}j\right)t} + e^{(-1/2 + \sqrt{3}/2\,j)t}\right]$$

case 2) Distinct poles of multiplicity k_i

Transfer function $G(s) = \dfrac{B(s)}{(s - \lambda_1)^{k_1}(s - \lambda_2)^{k_2} \cdots (s - \lambda_\ell)^{k_\ell}}$

n degree polynomial in the denominator: we expect n roots (poles)

$$= \frac{r_{1,1}}{s-\lambda_1} + \frac{r_{1,2}}{(s-\lambda_1)^2} + \cdots + \frac{r_{1,k_1}}{(s-\lambda_1)^{k_1}}$$
$$+ \underbrace{\frac{r_{2,2}}{(s-\lambda_2)^2}}_{\to \frac{r_{2,1}}{s-\lambda_2}} + \cdots + \frac{r_{2,k_2}}{(s-\lambda_2)^{k_2}}$$
$$+ \cdots + \frac{r_{\ell,1}}{s-\lambda_\ell} + \frac{r_{\ell,2}}{(s-\lambda_\ell)^2} + \cdots + \frac{r_{\ell,k_\ell}}{(s-\lambda_\ell)^{k_\ell}}$$

$$r_{i,k_i} = F(s)(s-\lambda_i)^{k_i}\Big|_{s=\lambda_i}$$

$$r_{i,k_i-j} = \frac{1}{j!}\frac{d^j}{ds^j}\left(F(s)(s-\lambda_i)^{k_i}\right)\Big|_{s=\lambda_i}$$

Example:

$$F(s) = \frac{1}{s^2(s+1)} \qquad \begin{cases} \lambda_1 = \lambda_2 = 0 & \to \text{multiplicity 2} \\ \lambda_3 = -1 & \to \text{multiplicity 1} \end{cases}$$
$$= \frac{r_1}{s} + \frac{r_2}{s^2} + \frac{r_3}{s+1}$$

orig. p. 8

$$r_2 = s^2 \frac{1}{s^2(s+1)}\Big|_{s=0} = 1$$

$$r_1 = \frac{d}{ds}\left(\frac{1}{s+1}\right)\Big|_{s=0} = \frac{-1}{(s+1)^2}\Big|_{s=0} = -1$$

$$r_3 = \underbrace{\frac{B(s)}{A'(s)}}_{\text{From case 1}}\Big|_{s=-1} = \underbrace{\frac{1}{3s^2+2s}}_{(s^3+s^2)'\text{nin türevi}}\Big|_{s=-1} = 1$$

case 3 proper ($n = m$) and non proper ($m > n$) T.F.S

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^{n1}}{a_0 + a_1 s + \cdots + a_n s^n}$$

* make at least some potion of T.F. strictly proper ($n > m$)

$$\frac{s^2+1}{5s^2+3} = \frac{1}{5}\left(\frac{s^2+1}{s^2+3/5}\right) = \frac{1}{5}\left(\frac{s^2+3/5-3/5+1}{s^2+3/5}\right)$$
$$= \frac{1}{5}\left(1 + \frac{2/5}{s^2+3/5}\right)$$

$$\frac{s^4+s^3-2s^2+1}{s^3+2s^2+s} \quad \text{(non proper)}$$
$$= \frac{s\left(s^3+s^2-2s\right)+1}{s^3+2s^2+s}$$
$$= \frac{s(s^3+2s^2+s) - s(2s^2+s) + s(s^2-2s) + 1}{s^3+2s^2+s}$$

[1] Claude notu: muhtemelen $b_m s^m$ olmalı.

$$= s + \frac{-2s^3 - s^2 + s^3 - 2s^2 + 1}{s^3 + 2s^2 + s} = s - \frac{s^3 + 3s^2 - 1}{s^3 + 2s^2 + s}$$

$$= s - \frac{(s^3 + 2s^2 + s) - 2s^2 - s + 3s^2 - 1}{s^3 + 2s^2 + s} = s - 1 + \underbrace{\frac{s^2 - s - 1}{s^3 + 2s^2 + s}}_{\hookrightarrow\, s(s^2+2s+1)}$$ [2]

$g(t) = \delta'(t) + \delta(t) + \dots$

[2] Claude notu: muhtemelen $s - 1 - \frac{s^2-s-1}{s^3+2s^2+s}$ olmalı.