Kontrol Sistemleri

State Space and 2nd Order Systems

Original notes: Control-Systems-lecture-5.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orig. p. 1

State Space

2^{nd} order D.E. $\xrightarrow{S.S.}$ 2 first order D.E.

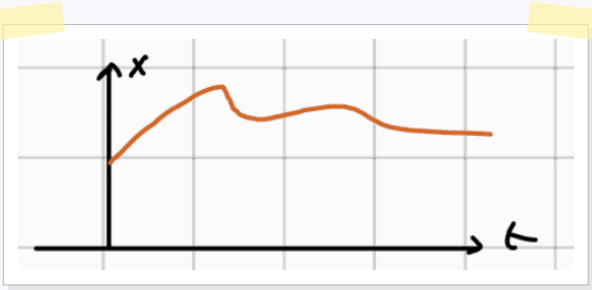
System of $\begin{cases} 2^{nd} \text{ order} \\ \text{equations} \\ (n) \end{cases}$ $\xrightarrow{S.S.}$ $2n$ first order D.E.

Example

$\ddot{x} + \alpha_1\dot{x} + \alpha_2 x = u$

x: output 2^{nd} order D.E.

u: input



original drawing, p. 1

$x = y_1$

$\dot{x} = y_2$

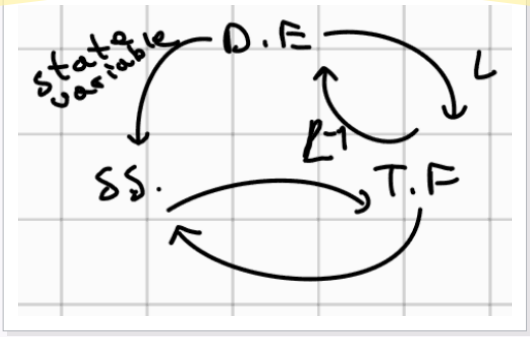
$\dot{y}_1 = \dot{x} = y_2$

$\dot{y}_2 = \ddot{x} = u - \alpha_1\dot{x} - \alpha_2 x$

$= u - \alpha_1 y_2 - \alpha_2 y_1$

y_1, y_2 (circled): state variables

$$\frac{d}{dt}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{state variable vector } (x)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{input vector } (B)}\underbrace{u}_{\text{output}^1}$$



original drawing, p. 1

$$\dot{\vec{X}} = A\vec{X} + \vec{B}u$$

Claude notu: muhtemelen input olmalı.

State space representation

↪ Fundamental operation (Mathematically) is two lower the order of D.E to half. If you are having $2n$ equation of 2^{nd} order this method produces $4n$ D.E. of 1^{st} order.

* States of a system are the min number of variables to know at $t = t_0$ in order to describe the dynamics (response) at any later time.

Example

$\ddot{y} + \alpha_1\dot{y} + \alpha_2 y = \dot{u} + \alpha_3 u$, y: output, u: input

States can be $y(0)$, $\dot{y}(0)$ → $y(t)$ = ✓

variables
- → output
- → input
- → state

orig. p. 2

State space representations

- → physical variable ✓
- → phase variable ✓
- → canonical variable
  - → observer ×
  - → controller ×

In physical variable form we just look at the energy storing elements

Capacitor: $\frac{1}{2}Cv^2$ Voltage: state variable

Inductor: $\frac{1}{2}Li^2$ Current "

mass: $\frac{1}{2}I\omega^2$ / $\frac{1}{2}mv^2$ ω / v : state variable

spring: $\frac{1}{2}kx^2$

Phase variable form

→ Case 1: No derivative of the input

$$\dot{u} = \frac{du}{dt} \begin{cases} \rightarrow \text{forward difference} & \lim_{\Delta t \to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t}\Big|_{t=0} \\ \rightarrow \text{backward difference} & \lim_{\Delta t \to 0} \frac{u(t) - u(t-\Delta t)}{\Delta t} \quad t = -\Delta t \\ \rightarrow \text{central difference} & \lim_{\Delta t \to 0} \frac{u\left(t+\frac{\Delta t}{2}\right) - u\left(t-\frac{\Delta t}{2}\right)}{\Delta t} \end{cases}$$

$\dot{y} = u$

$\Downarrow$

y: output

u: input

$\ddot{y} = \ddot{u}$

y: output

u: input

for $\ddot{y} = \ddot{u} - \dot{y}$

you have to know $y(0), \dot{y}(0)$ ok.

and $\ddot{u}$ is given

$$\overset{\text{order}}{y^{(n)}} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = u$$

$$\text{state variables} \begin{cases} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \\ \vdots \\ x_n = y^{(n-1)} \end{cases} \xrightarrow{\frac{d}{dt}} \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = x_3 \\ \vdots \end{cases}$$

$$\dot{x}_n = y^{(n)} = u - a_1 \overset{x_n}{y^{(n-1)}} - a_2 \overset{x_{n-1}}{y^{(n-2)}} - \cdots - a_n \overset{x_1}{y}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 \\ \vdots & & & & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

orig. p. 3

$$\dot{\vec{X}} = A_{n\times n}\vec{X}_{n\times 1} + B_{n\times 1}u$$

$$\dot{\vec{X}} = A\vec{X} + Bu \rightarrow \text{state space representation}$$

$$y_{1\times1} = \underbrace{\begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix}}_{C} \underbrace{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}}_{X} = C\vec{X} + \overset{0}{\cancel{D}}u$$

A: state matrix
B: input vector
C: output vector
D: Direct-?transmission matrix/vector

→ Case 2: Derivative of input is present

$$\overbrace{\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3y}^{\text{LHS}} = \overbrace{b_0\ddot{u} + b_1\dot{u} + b_2\dot{u} + b_3u}^{\text{RHS}}$$

RHS → statically related (can be chosen) u: input, y: output

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \end{cases} \rightarrow \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = x_3 \\ \dot{x}_3 = \dddot{y} = \text{RHS} - \text{LHS} \end{cases}$$

LHS: left hand side
RHS: right hand side

Burdan SS. formatına sokamayız
O yüzden bunu yapalım

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_ny = b_0u^{(n)} + b_1u^{(n-1)} + \cdots + b_nu$$

u: input
y: output number of state variables n

ezberlemeye gerek yok, gerekirse sınavda verilir

$$\begin{cases} x_1 = y - \beta_0u \\ x_2 = \dot{y} - \beta_0\dot{u} - \beta_1u = \dot{x}_1 - \beta_1u \\ x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2u = \dot{x}_2 - \beta_2u \\ \vdots \\ x_n = \dot{x}_{n-1} - \beta_{n-1}u \end{cases}$$

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1\beta_0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\ &\vdots \\ \beta_{n-1} &= b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0 \end{aligned}$$

orig. p. 4

Phase variable form

$$\begin{cases} \dot{x}_1 = \dot{y} - \beta_0 \dot{u} = x_2 + \beta_1 u \\ \dot{x}_2 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = x_3 + \beta_2 u \\ \vdots \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ -a_n & -a_{n-1} & -a_{n-2} & & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u$$

$$\dot{\vec{X}} = A\vec{X} + Bu$$

$$y = x_1 + \beta_0 u = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + (\beta_0) u$$

$$y = C\vec{X} + Du$$

$$\text{T.F} = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Example

$$\overbrace{\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y}^{\text{LHS}} = \overbrace{b_0 \ddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u}^{\text{RHS}}$$

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1 \beta_0 \\ \beta_2 &= b_2 - a_1 \beta_1 - a_2 \beta_0 \\ \beta_3 &= b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0 \end{aligned}$$

$$\begin{aligned} x_1 &= y - b_0 u \\ x_2 &= \dot{y} - b_0 \dot{u} - (b_1 - a_1 \beta_0) u \\ x_3 &= \ddot{y} - b_0 \ddot{u} - (b_1 - a_1 \beta_0) \dot{u} - (b_2 - a_1 \beta_1 - a_2 \beta_0) u \end{aligned}$$

orig. p. 5

Figure: Top of the page: a pasted, low-contrast photograph of a handwritten derivation; its legible parts are transcribed below. *(in the original notes, p. 5)*

$$\begin{cases} x_1 = y - \beta_0 u \\ x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \end{cases} \rightarrow \begin{cases} \dot{x}_1 = \dot{y} - \beta_0 \dot{u} = x_2 + \beta_1 u \\ \dot{x}_2 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = x_3 + \beta_2 u \end{cases}$$

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1 \beta_0 \\ \beta_2 &= b_2 - a_1 \beta_1 - a_2 \beta_0 \end{aligned}$$

$$\begin{aligned} \dot{x}_3 &= \dddot{y} - \beta_0 \dddot{u} - \beta_1 \ddot{u} - \beta_2 \dot{u} \\ &= (-a_1 \ddot{y} - a_2 \dot{y} - a_3 y + b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u) \\ &\quad - \beta_0 \dddot{u} - \beta_1 \ddot{u} - \beta_2 \dot{u} \\ &= \Big[- a_1 (\ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u) - a_1 \beta_0 \ddot{u} - a_1 \beta_1 \dot{u} - a_1 \beta_2 u \\ &\quad - a_2 (\dot{y} - \beta_0 \dot{u} - \beta_1 u) - a_2 \beta_0 \dot{u} - a_2 \beta_1 u \\ &\quad - a_3 (y - \beta_0 u) - a_3 \beta_0 u \Big] \\ &\quad + b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u - \beta_0 \dddot{u} - \beta_1 \ddot{u} - \beta_2 \dot{u} \\ &= -a_1 x_3 - a_2 x_2 - a_3 x_1 + \ddot{u} \underbrace{(b_0 - \beta_0)}_{0} + \ddot{u} \underbrace{(b_1 - \beta_1 - a_1 \beta_0)}_{0} \\ &\quad + \dot{u} \underbrace{(-a_1 \beta_1 - a_2 \beta_0 + b_2 - \beta_2)}_{0} + u \underbrace{(-a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0 + b_3)}_{\beta_3} \end{aligned}$$

$$\begin{cases} \dot{x}_1 = x_2 + \beta_1 u \\ \dot{x}_2 = x_3 + \beta_2 u \\ \dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + \beta_3 u \end{cases} \qquad y = x_1 + \beta_0 u$$

$$\dot{\vec{X}} = A\vec{X} + Bu \rightarrow \text{time domain}$$

$$y = C\vec{X} + Du$$

$\mathcal{L}$: S.S. $\rightarrow$ TF

$$sX(s) - X(0) = AX(s) + BU(s) \rightarrow (sI - A)\vec{X}(s) = BU(s)$$

$$X(s) = (sI - A)^{-1} BU(s) \rightarrow Y(s) = CX(s) + DU(s)$$

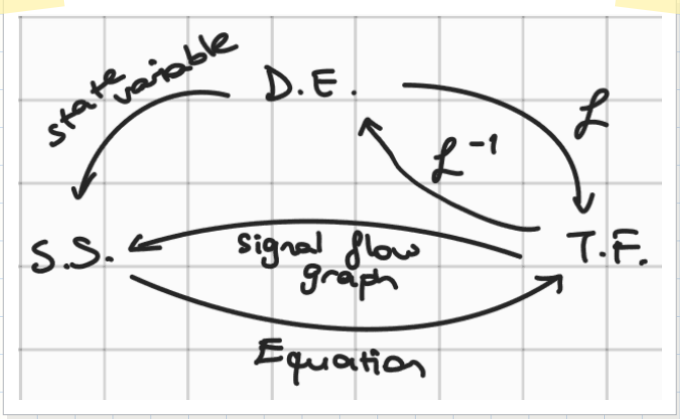
$$Y(s) = C(sI - A)^{-1} BU(s) + DU(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

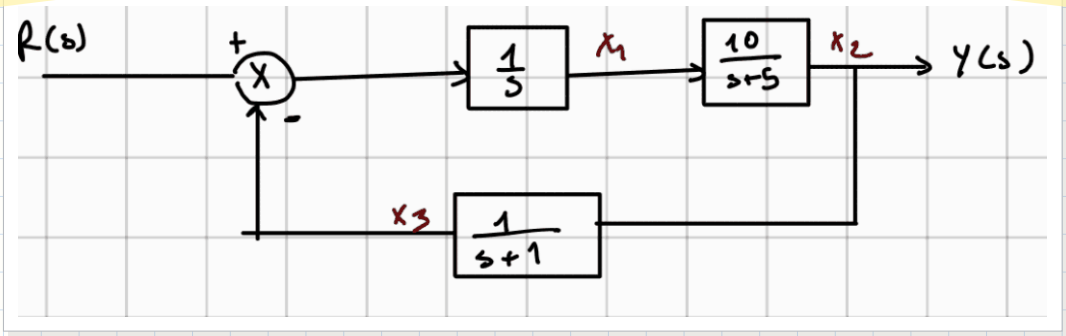
$\hookrightarrow$ a vector of T.F

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

orig. p. 6



original drawing, p. 6



original drawing, p. 6

$$\frac{Y(s)}{X(s)} = \frac{1}{s} \rightarrow \text{integration} \qquad \frac{Y(s)}{X(s)} = s \rightarrow \text{derivative}$$

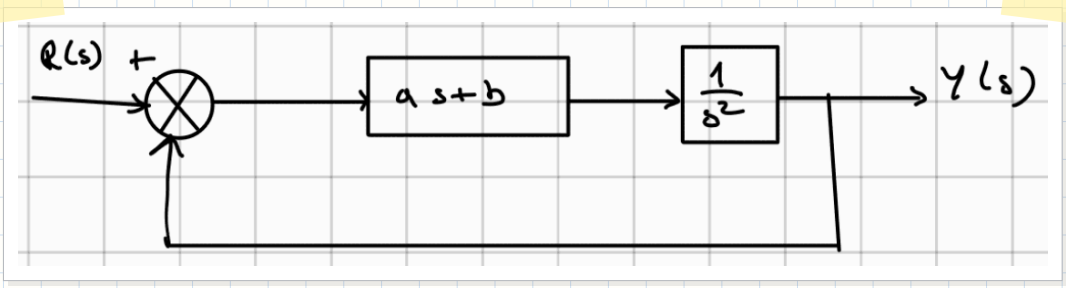
$$\frac{x_3}{x_2} = \frac{1}{s+1}, \qquad \frac{x_2}{x_1} = \frac{10}{s+5}, \qquad \frac{x_1}{R - x_3} = \frac{1}{s}$$

$$x_2 = sx_3 + x_3 \Rightarrow sx_3 = x_2 - x_3 \Rightarrow \dot{x}_3 = x_2 - x_3$$

$$10x_1 = sx_2 + 5x_2 \Rightarrow sx_2 = 10x_1 - 5x_2 \Rightarrow \dot{x}_2 = 10x_1 - 5x_2$$

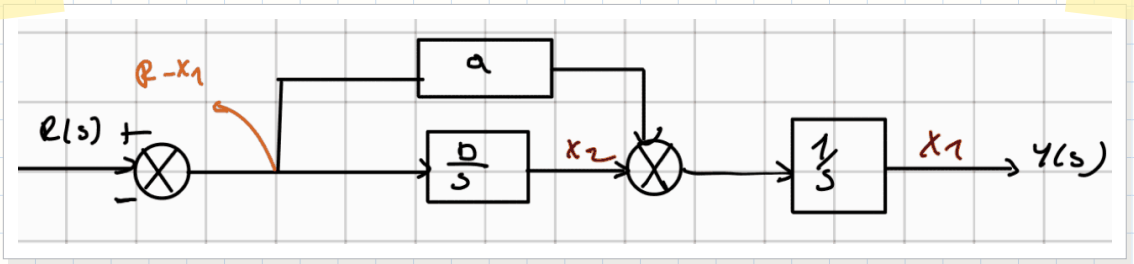
$$sx_1 = R - x_3 \Rightarrow \dot{x}_1 = r - x_3$$

$$\left.\begin{aligned} \dot{x}_1 &= -x_3 + r \\ \dot{x}_2 &= 10x_1 - 5x_2 \\ \dot{x}_3 &= x_2 - x_3 \end{aligned}\right\} \rightarrow \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 10 & -5 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} r$$



original drawing, p. 6

$$\frac{as+b}{s^2} = \frac{a}{s} + \frac{b}{s^2} = \frac{1}{s}\left(a + \frac{b}{s}\right)$$



original drawing, p. 6

$$\frac{x_2}{R - x_1} = \frac{b}{s} \quad (1) \qquad sx_2 = b(R - x_1)$$

$$\frac{x_1}{x_2 + a(R - x_1)} = \frac{1}{s} \quad (2) \qquad sx_1 = x_2 + a(R - x_1)$$

orig. p. 7

Example

$m_1\ddot{y}_1 = b_1(\dot{y}_2 - \dot{y}_1) - k_1 y_1 + u_1$

$m_2\ddot{y}_2 = b_2(\dot{y}_2 - \dot{y}_1) - k_2 y_2 + u_2$

m_1, m_2 : mass values
b_1 : damping value
k_1, k_2 : stiffness values
} these are all numbers

u_1, u_2: inputs, y_1, y_2: outputs

S.S.

$x_1 = y_1$ $\qquad \dot{x}_1 = \dot{y}_1 = x_2$

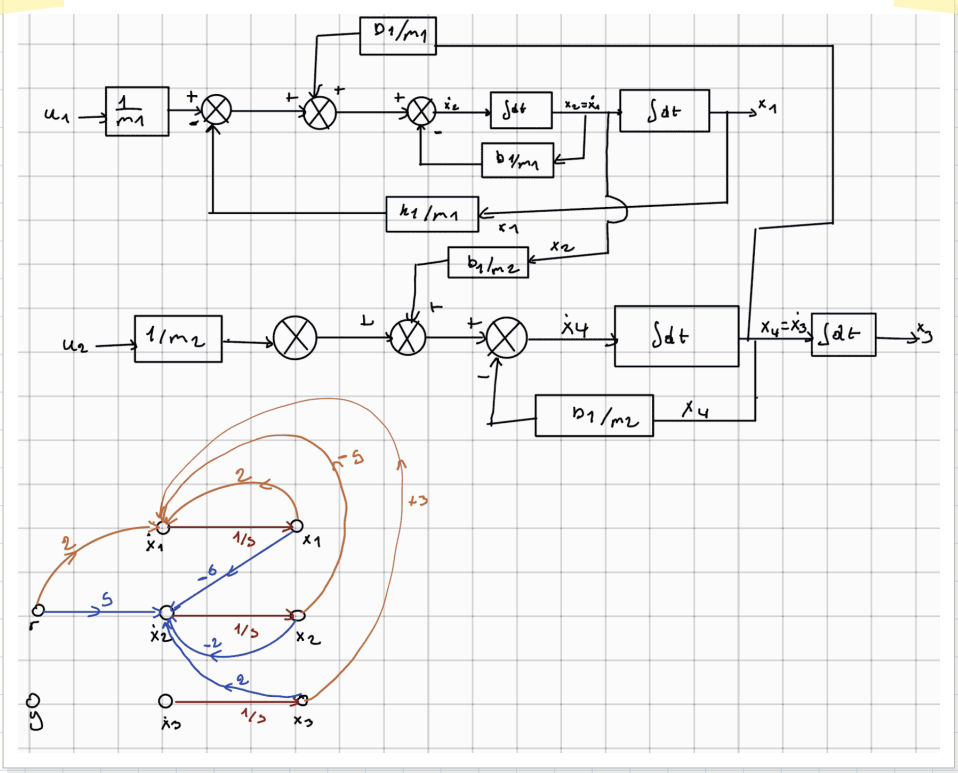
$x_2 = \dot{y}_1 \Rightarrow$ $\qquad \dot{x}_2 = \ddot{y}_1 = \frac{1}{m_1}\left[b_1(\dot{y}_2 - \dot{y}_1) - k_1 y_1 + u_1\right]$

$= \frac{1}{m_1}\left[b_1(x_4 - x_2) - k_1 x_1 + u_1\right]$

$x_3 = y_2 \Rightarrow$ $\qquad \dot{x}_3 = \dot{y}_2 = x_4$

$x_4 = \dot{y}_2$ $\qquad \dot{x}_4 = \ddot{y}_2 = \frac{1}{m_2}\left[-b_1(\dot{y}_2 - \dot{y}_1) - k_2 y_2 + u_2\right]$

$= \frac{1}{m_2}\left[-b_1(x_4 - x_2) - k_2 x_3 + u_2\right]$



original drawing, p. 7

orig. p. 8

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & 0 & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{b_1}{m_2}\,2 & -\frac{k_2}{m_2} & -\frac{b_1}{m_2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{B\ (2\times 4^3)} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\vec{u}\ (2\times 1)}$$

$$sX_2 = bR - X_1 b \qquad \dot{x}_2 = br - x_1 b$$

$$\dot{x}_1 = x_2 + a(r - x_1)$$

$$sX_1 = X_2 + a(R - X_1)$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix}}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b \\ a \end{bmatrix}^4 \begin{bmatrix} r \end{bmatrix}$$

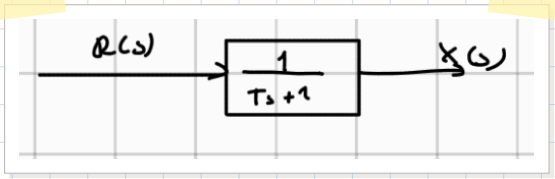
$$\dot{\vec{x}} = A\vec{x} + Bu$$

$$y = Cx + Du$$

[2] Claude notu: $\dot{x}_4$ denklemine (s. 7) göre A matrisinin (4,2) elemanı muhtemelen $+b_1/m_2$ olmalı.
[3] Claude notu: B matrisinin boyutu muhtemelen 4 × 2 olmalı.
[4] Claude notu: yukarıdaki denklemlere göre muhtemelen $\begin{bmatrix} a \\ b \end{bmatrix}$ olmalı.

1st order system

III) Ramp Response



original drawing, p. 8

$$\begin{cases} r(t) = t \\ R(s) = \dfrac{1}{s^2} \end{cases}$$

$$X(s) = R(s)\frac{1}{Ts+1} = \frac{1}{s^2(Ts+1)}$$

$$= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{Ts+1}$$

$$s_{1,2} = 0$$
$$s_3 = -\frac{1}{T}$$

$$A = s^2\left(\frac{1}{s^2(Ts+1)}\right)\Big|_{s=0} = 1$$

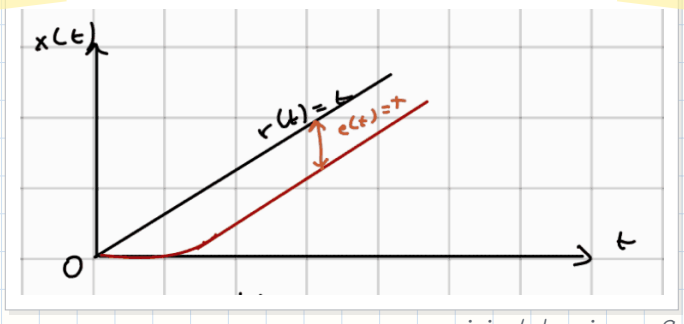
$$B = \frac{d}{ds}\left(s^2\frac{1}{s^2(Ts+1)}\right)\Big|_{s=0} = -T$$

$$C = (Ts+1)\frac{1}{s^2(Ts+1)}\Big|_{s=-\frac{1}{T}} = T^2$$

orig. p. 9

$$X(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1} = \frac{T}{s+\frac{1}{T}}$$

$$\mathcal{L}^{-1}\{\ x(t) = t - T + Te^{-t/T}$$



original drawing, p. 9

$$\hat{x} = 1 - e^{-t/T}$$

$$e(t) = r(t) - x(t)$$
$$= t - \left[t - T + Te^{-t/T}\right]$$
$$= T - Te^{-t/T}$$

$$@\ t = 0 \longrightarrow e(0) = 0$$
$$t \uparrow^{\infty} \Rightarrow e(\infty) = e_{ss} = T$$

Accuracy is defined by 'T'.

Ramp response = $\int$ unit response = $\iint$ impulse response

$$\frac{d}{dt}(\text{ramp resp.}) = \text{step resp.}$$
$$\frac{d^2}{dt^2}(\text{ramp resp.}) = \text{impulse resp.}$$

2nd order response

RLC in series

$$\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

RLC in parallel

$$\frac{\Phi(s)}{I(s)} = \frac{1}{Cs^2 + \frac{s}{R} + \frac{1}{L}} = \frac{1/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Spring mass Damper System

$$\frac{X(s)}{R(s)} = \frac{1}{ms^2 + cs + k} = \frac{1/m}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

$$\Rightarrow \frac{\text{output}}{\text{input}} = \frac{X(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n = natural frequency

ζ = zeta = damping ratio

orig. p. 10

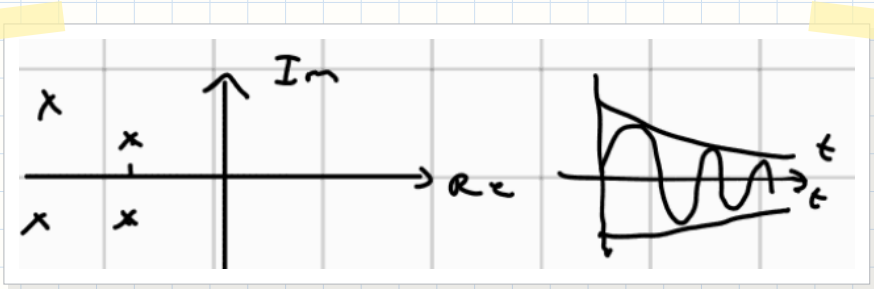
no zeros but 2 poles are present

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Underdamped Systems

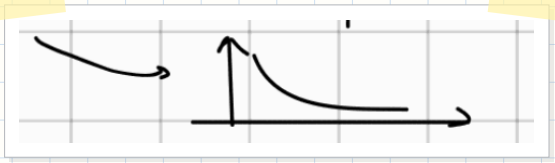
if $\zeta < 1 \Rightarrow$ complex conjugate poles *(always oscillatory)*



original drawing, p. 10

critically damped systems

(sorry) if $\zeta = 1 \Rightarrow$ repeated real valued poles



original drawing, p. 10

if $\zeta > 1 \Rightarrow$ seperate and real valued poles overdamped system

Homework (homogen solution)

Response due to external excitation (particular solution)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Specify the input

I) Impulse excitation $R(s) = 1$

case I) $\zeta < 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n - i\omega_d)(s + \zeta\omega_n + i\omega_d)}$$
$$= \frac{A}{s + \zeta\omega_n - i\omega_d} + \frac{B}{s + \zeta\omega_n + i\omega_d}$$

$$s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$
$$= \omega_n\left(-\zeta \pm i\sqrt{1 - \zeta^2}\right)$$
$$= -\zeta\omega_n \pm i \underbrace{\omega_n\sqrt{1 - \zeta^2}}_{\text{damped natural freq.} = \omega_d}$$

The response frequency for damped systems is ω_d

$$\frac{C(s)}{R(s)} = \frac{A}{s+\zeta\omega_n - i\omega_d} + \frac{B}{s+\zeta\omega_n + i\omega_d} \Rightarrow c(t)$$

$$A = \left.\frac{\omega_n^2}{s+\zeta\omega_n + i\omega_d}\right|_{s=-\zeta\omega_n+i\omega_d}$$

orig. p. 11

$$B = \left.\left(\frac{\omega_n^2}{s+\zeta\omega_n - i\omega_d}\right)\right|_{s=-\zeta\omega_n-i\omega_d}$$

$$\begin{aligned} c(t) &= A e^{(-\zeta\omega_n+i\omega_d)t} + B e^{(-\zeta\omega_n-i\omega_d)t} \\ &= e^{-\zeta\omega_n t}\left(A e^{i\omega_d t} + B e^{-i\omega_d t}\right) \\ &= a\, e^{-\zeta\omega_n t}\cos\omega_d t \end{aligned}$$



original drawing, p. 11

$$\begin{cases} e^{i\alpha} = \cos\alpha + i\sin\alpha \\ e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \longrightarrow \frac{e^{i\alpha}+e^{-i\alpha}}{2} = \cos\alpha$$

case II) $\zeta = 1$; $s_{1,2} = -\zeta\omega_n$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n)^2} = \frac{\hat{A}}{s+\zeta\omega_n} + \frac{\hat{B}}{(s+\zeta\omega_n)^2}$$

$$\hat{B} = (s+\zeta\omega_n)^2\left.\left(\frac{\omega_n^2}{(s+\zeta\omega_n)^2}\right)\right|_{s=-\zeta\omega_n}$$

$$\hat{A} = \frac{d}{ds}\left.\left(\frac{\omega_n^2}{s+\zeta\omega_n}\right)\right|_{s=-\zeta\omega_n}$$

critical damping

$$\begin{aligned} c(t) &= \hat{A}\, e^{-\zeta\omega_n t} + \hat{B}\, t\, e^{-\zeta\omega_n t} \\ &= e^{-\zeta\omega_n t}\left(\hat{A} + \hat{B}t\right) \end{aligned}$$

? case III) $(\zeta > 1)$

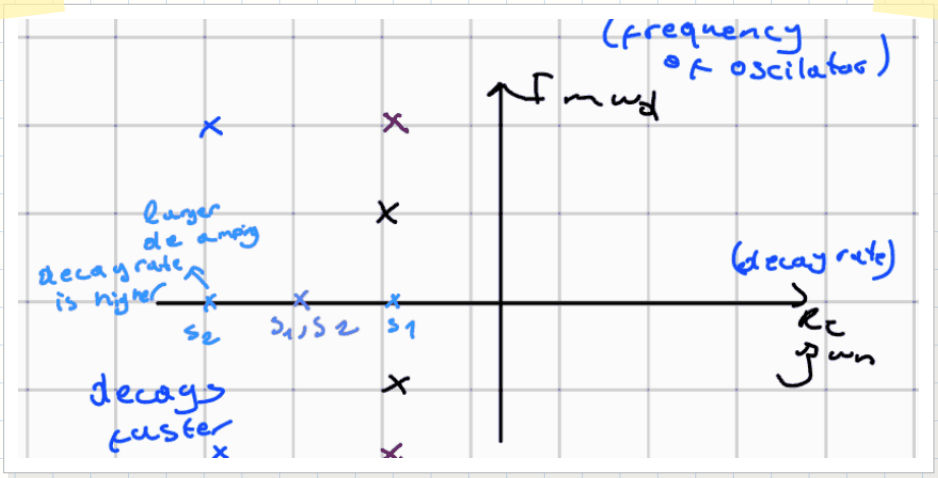
$$c(t) = \hat{A}\, e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} + \bar{B}\, e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$$

2nd order systems

$\zeta < 1 \quad s_{1,2} = \underbrace{-\zeta\omega_n}_{\text{Re}} \pm \underbrace{i\omega_d}_{\text{Im}}$

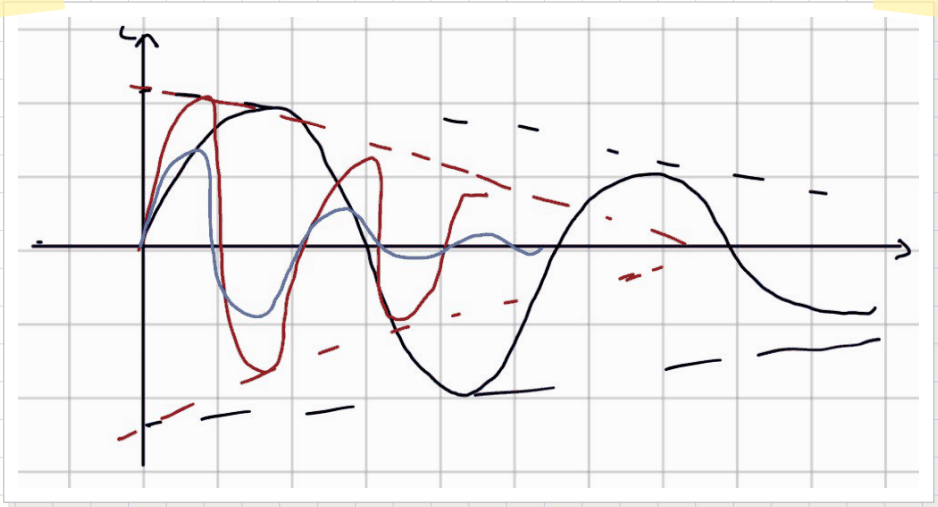
$\zeta = 1 \quad s_{1,2} = \underbrace{-\zeta\omega_n}_{\text{repeated}} \quad \text{Re}$

$\zeta > 1 \quad s_{1,2} = \underbrace{-\zeta\omega_n \pm \sqrt{\zeta^2 - 1}}_{\text{Re distinct}}$[5]



original drawing, p. 11

orig. p. 12 If poles are C.C. → response is oscillatory and decaying



original drawing, p. 12

for ?x $\omega_d = 0$ no oscillating frequency

?decays at the same time

[5] Claude notu: muhtemelen $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ olmalı.

for $\zeta > 1$

$$s_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}$$
$$s_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}$$

$$c(t) = \underbrace{Ae^{s_1 t}}_{\text{this is the most important part}} + Be^{s_2 t} \quad \searrow \text{ decay fastly}$$

Step response calculation

(just for $\zeta < 1$)

$$\begin{cases} \dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ R(s) = \dfrac{1}{s} \end{cases}$$

$$\begin{aligned} C(s) &= \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \end{aligned}$$

? $\dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_d$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$

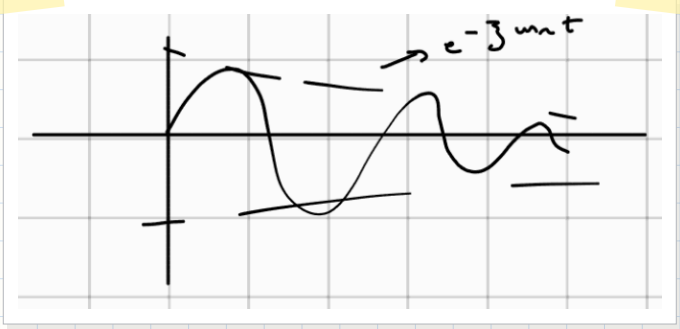
$$\mathcal{L}\left\{e^{-at}\sin\omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2}$$
$$\mathcal{L}\left\{e^{-at}\sin\omega t\right\}^{6} = \frac{s+a}{(s+a)^2 + \omega^2}$$

orig. p. 13

$$\begin{aligned} c(t) = \mathcal{L}^{-1}\{C(s)\} &= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\omega_d t \\ &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \end{aligned}$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

[6] Claude notu: muhtemelen $\cos\omega t$ olmalı.



original drawing, p. 13

$$e(t) = r(t) - c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \phi\right)$$