Kontrol Sistemleri

Transient / Steady State Response (1st & 2nd order)

Original notes: Control-Systems-lecture-6.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orig. p. 1

Transient / Steady State Response

We may check the responses for 3 characteristics:

1- Quality of response
2- Accuracy of response (% error)
3- Stability

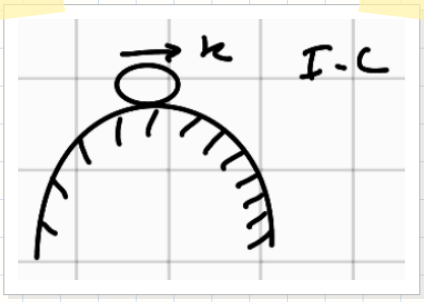
Differential equations:

$$\text{1st:} \quad T\dot{x} + x = r$$
$$\text{2nd:} \quad \ddot{x} + a_1\dot{x} + a_2x = r$$

$$x(t) = x_h(t) + x_p(t)$$

x_h: homogenous solution, due to I.C $\Rightarrow$ transient response; x_p: particular solution, due to external excitation.

To check stability we have to check the homogen solution.

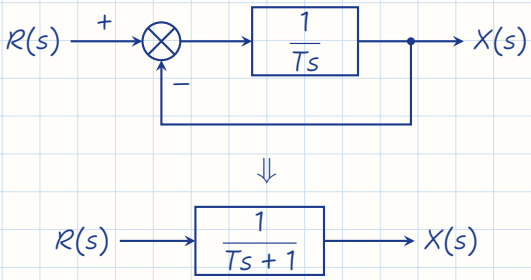


original drawing, p. 1

or it can be unstable due to resonans, $r = \varepsilon \sin \omega t$.

$$\text{To get } x_h(t) \rightarrow \begin{cases} T\dot{x} + x = 0 \\ \ddot{x} + a_1\dot{x} + a_2x = 0 \end{cases}$$

First order system



$$T\frac{dx}{dt} + x = r \qquad \text{initial condition } x(0) = x_0$$

$$RC\frac{dv_c}{dt} + v_c = V \qquad T\dot{v}_c + v_c = V \Rightarrow \underbrace{(Ts+1)}_{\text{characteristic equation}} \qquad V_c = V$$

$$\mathcal{L} \Rightarrow T[sX(s) - x_0] + X = R \rightarrow X(s)(Ts+1) = R(s) + Tx(0)$$

$$\rightarrow X(s) = \underbrace{\frac{R(s)}{Ts+1}}_{\text{particular solut.}} + \underbrace{\frac{Tx(0)}{Ts+1}}_{\text{homo sol.}}$$

orig. p. 2

if $R(s) = 0 \rightarrow$ transient solution

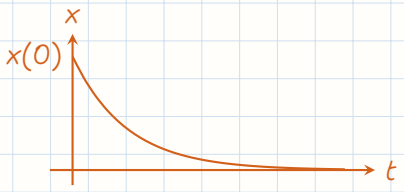
$T = RC$

$$X(s) = \frac{Tx(0)}{Ts+1} = Tx(0)\left(\frac{1/T}{s+1/T}\right) = \frac{x(0)}{s+\frac{1}{T}}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad x(t) = x_0 e^{-t/T} = x_0 e^{s_1 t}$$

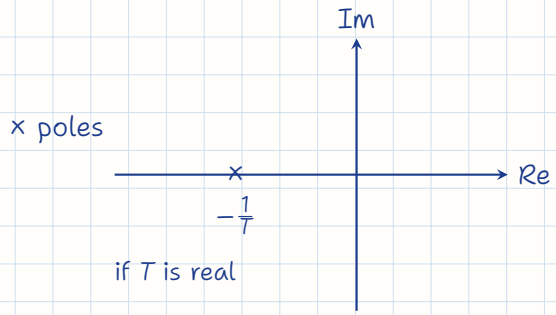
(s_1: position of pole determines the rate of decay.)

$$x(t) = \frac{x(0)}{e^{t/T}} = \frac{x(0)}{e^{|s_1|t}} \qquad |s_1| \uparrow \Rightarrow x(t) \text{ decreases faster}$$



T: time constant of the system

$$\text{characteristic equation}: \; Ts+1 \Rightarrow s_1 = -\frac{1}{T} \quad \text{(pole of the system)}$$

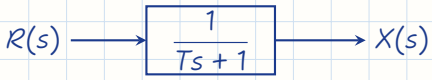


$T \uparrow \Rightarrow |s_1| \downarrow \Rightarrow$ closer to imaginary axis

$T \downarrow \Rightarrow |s_1| \uparrow \Rightarrow$ further from "

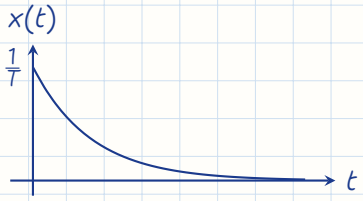
First order systems

Assume that there is no initial condition and $R(s) \neq 0$.



case 1) impulse input: $R(s) = 1$

$$X(s) = \frac{1}{Ts+1} = \frac{1/T}{s+\frac{1}{T}} \Rightarrow x(t) = \frac{1}{T} e^{-t/T} = |s_1| e^{-|s_1|t}$$



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case II) step excitation

$$r(t) = 1, \quad t \geq 0$$

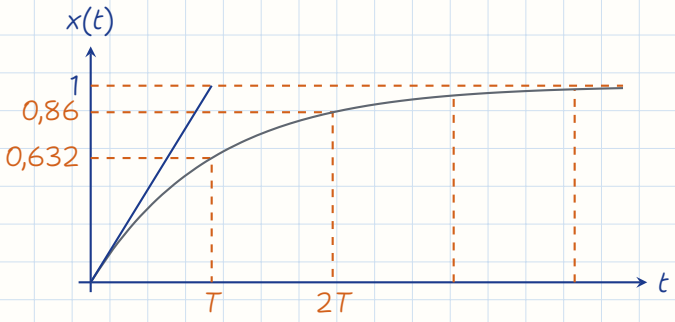
$$R(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(Ts+1)} = \frac{A}{s} + \frac{B}{Ts+1} = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad x(t) = 1 - e^{-t/T}$$

$$A = \left.\frac{s}{s(Ts+1)}\right|_{s=0} = 1$$

$$B = \left.\frac{Ts+1}{s(Ts+1)}\right|_{s=-\frac{1}{T}} = -T$$



at $t = 0 \rightarrow x(0) = 0$

t	$x(t)$	
T	0,632	63% of the final value
$2T$	0,86	
$3T$	0,95	
$4T$	0,98	it have to be reasonable setting time

$$e(t) = r(t) - x(t) = 1 - 1 + e^{-t/T} = e^{-t/T}$$

derivative of response gives us the rate of change (slope of response)

$$\text{slope} = \frac{dx(t)}{dt} = \frac{1}{T} e^{-t/T}$$

at $t = 0$, we get the max slope: $\left(\frac{1}{T}\right)$

orig. p. 4

$$T\dot{x} + x = r$$

only x has to be continuous

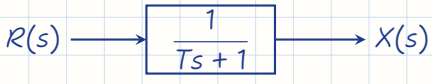
$$\ddot{x} + a_1\dot{x} + a_2 x = r$$

x and $\dot{x}$ has to be continuous

Impulse response = $\frac{d}{dt}$ (step response)

1st order system

III) Ramp Response



$$r(t) = t \rightarrow R(s) = \frac{1}{s^2}$$

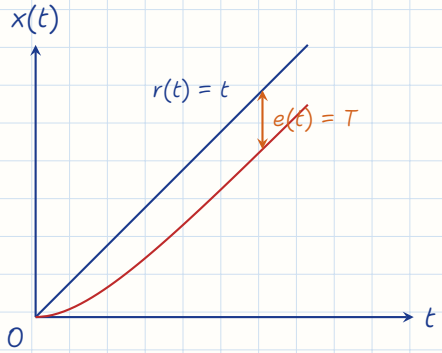
$$X(s) = R(s)\frac{1}{Ts+1} = \frac{1}{s^2(Ts+1)} \qquad \left(s_{1,2} = 0,\ s_3 = -\frac{1}{T}\right)$$

$$= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{Ts+1}$$

$$A = s^2\left(\frac{1}{s^2(Ts+1)}\right)\Big|_{s=0} = 1$$
$$B = \frac{d}{dt}^1\left(s^2\frac{1}{s^2(Ts+1)}\right)\Big|_{s=0} = -T$$
$$C = (Ts+1)\frac{1}{s^2(Ts+1)}\Big|_{s=-\frac{1}{T}} = T^2$$

orig. p. 5

$$X(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1} \quad \left[= \frac{T}{s+\frac{1}{T}}\right]$$
$$\xrightarrow{\mathcal{L}^{-1}} \quad x(t) = t - T + Te^{-t/T}$$



$$\dot{x} = 1 - e^{-t/T}$$

$$e(t) = r(t) - x(t)$$
$$= t - \left(t - T + Te^{-t/T}\right)$$
$$= T - Te^{-t/T}$$

at $t = 0 \rightarrow e(0) = 0$

$t \rightarrow \infty \Rightarrow e(\infty) = e_{ss} = T$

Accuracy is defined by 'T'.

$$\text{Ramp response} = \int \text{unit response} = \iint \text{impulse response}$$

$$\frac{d}{dt}(\text{ramp resp}) = \text{step resp}$$
$$\frac{d^2}{dt^2}(\text{ramp resp}) = \text{impulse resp}$$

2nd order response

RLC in series

$$\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

[1] Claude notu: muhtemelen d/ds olmalı.

RLC in parallel

$$\frac{\Phi(s)}{I(s)} = \frac{1}{Cs^2 + \frac{s}{R} + \frac{1}{L}} = \frac{1/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Spring mass Damper System

$$\frac{X(s)}{R(s)} = \frac{1}{ms^2 + cs + k} = \frac{1/m}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

$$\Rightarrow \frac{\text{output}}{\text{input}} = \frac{X(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n = natural frequency; ζ = zeta = damping ratio

orig. p. 6

no zeros but 2 poles are present

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\begin{aligned} s_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \end{aligned}$$

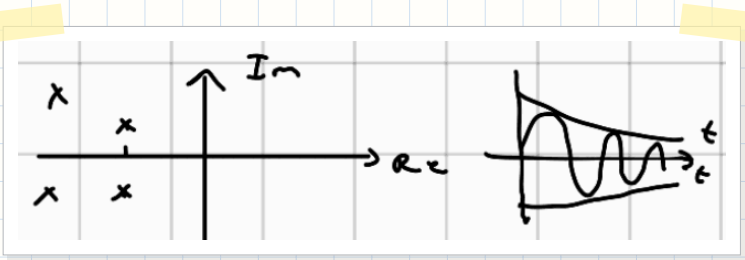
Underdamped systems

if $\zeta < 1 \Rightarrow$ complex conjugate poles *(always oscillatory)*

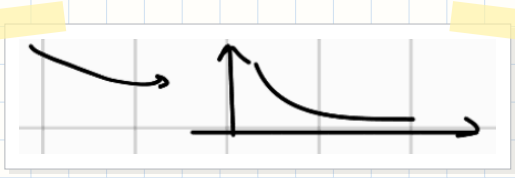
critically damped systems

(sorry) if $\zeta = 1 \Rightarrow$ repeated real valued poles

if $\zeta > 1 \Rightarrow$ seperate and real valued poles overdamped system



original drawing, p. 6



original drawing

Homework (homogen solution)

Response due to external excitation (particular solution)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Specifying the input

I) Impulse excitation $\quad R(s) = 1$

case I) $\zeta < 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n - i\omega_d)(s + \zeta\omega_n + i\omega_d)}$$
$$= \frac{A}{s + \zeta\omega_n - i\omega_d} + \frac{B}{s + \zeta\omega_n + i\omega_d}$$

$$s_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$$
$$= \omega_n(-\zeta \pm i\sqrt{1 - \zeta^2})$$
$$= -\zeta\omega_n \pm i \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\substack{\text{damped natural freq} \\ = \omega_d}}$$

The response frequency for damped systems is ω_d

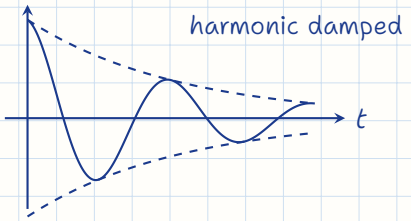
$$\frac{C(s)}{R(s)} = \frac{A}{s + \zeta\omega_n - i\omega_d} + \frac{B}{s + \zeta\omega_n + i\omega_d} \Rightarrow c(t)$$

$$A = \left.\frac{\omega_n^2}{s + \zeta\omega_n + i\omega_d}\right|_{s=-\zeta\omega_n+i\omega_d}$$

orig. p. 7

$$B = \left.\left(\frac{\omega_n^2}{s + \zeta\omega_n - i\omega_d}\right)\right|_{s=-\zeta\omega_n-i\omega_d}$$

$$c(t) = A e^{(-\zeta\omega_n+i\omega_d)t} + B e^{(-\zeta\omega_n-i\omega_d)t}$$
$$= e^{-\zeta\omega_n t}(A e^{i\omega_d t} + B e^{-i\omega_d t})$$
$$= \alpha e^{-\zeta\omega_n t}\cos\omega_d t$$



$$\begin{cases} e^{i\alpha} = \cos\alpha + i\sin\alpha \\ e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \rightarrow \frac{e^{i\alpha} + e^{-i\alpha}}{2} = \cos\alpha$$

case II) $\zeta = 1$; $\quad s_{1,2} = -\zeta\omega_n$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2} = \frac{\hat{A}}{s + \zeta\omega_n} + \frac{\hat{B}}{(s + \zeta\omega_n)^2}$$

$$\hat{B} = (s+\zeta\omega_n)^2 \left(\frac{\omega_n^2}{(s+\zeta\omega_n)^2}\right)\Big|_{s=-\zeta\omega_n}$$

$$\hat{A} = \frac{d}{ds}\left(\frac{\omega_n^2}{s+\zeta\omega_n}\right)\Big|_{s=-\zeta\omega_n}$$

critical damping

$$\begin{aligned} c(t) &= \hat{A}e^{-\zeta\omega_n t} + \hat{B}te^{-\zeta\omega_n t} \\ &= e^{-\zeta\omega_n t}(\hat{A}+\hat{B}t) \end{aligned}$$

III) $(\zeta > 1)$

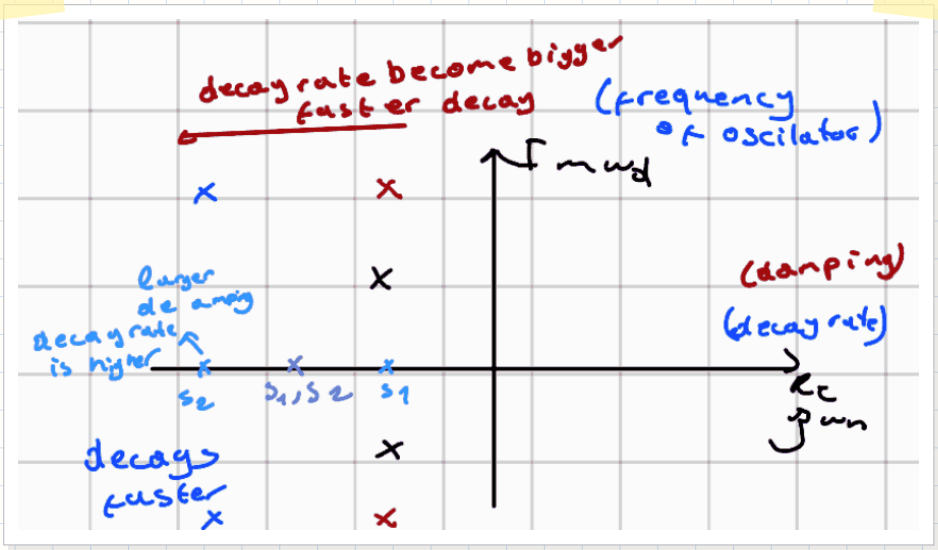
$$c(t) = \hat{A}e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} + \hat{B}e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t}$$

2nd order systems

$$\zeta < 1: \quad s_{1,2} = \underbrace{-\zeta\omega_n}_{\text{Re}} \pm \underbrace{i\omega_d}_{\text{Im}}$$

$$\zeta = 1: \quad s_{1,2} = \underbrace{-\zeta\omega_n}_{\text{repeated}} \quad \text{Re}$$

$$\zeta > 1: \quad s_{1,2} = \underbrace{-\zeta\omega_n}_{\text{Re distinct}} \pm \sqrt{\zeta^2-1}^2$$

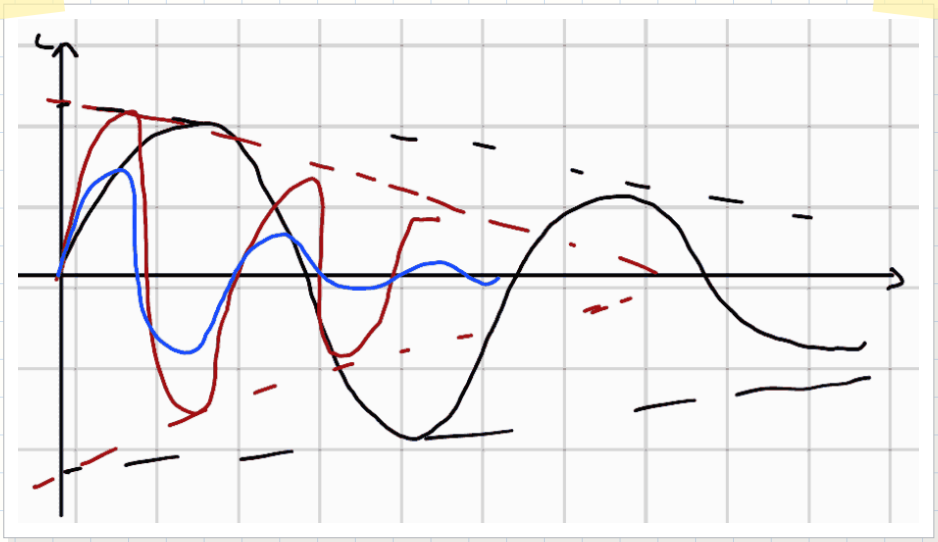


original drawing, p. 7

orig. p. 8

If poles are CC → response is oscillatory and decaying

[2] Claude notu: muhtemelen $\pm\omega_n\sqrt{\zeta^2-1}$ olmalı.



original drawing, p. 8

for × $\omega_d = 0$ no oscillating frequency; decays at the same time

for $\zeta > 1$

$$s_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}$$

$$c(t) = \underbrace{A e^{s_1 t}}_{\text{this is the most important part}} + B e^{s_2 t} \rightarrow \text{decay fastly}$$

Step response calculation

(just for $\zeta < 1$)

$$\begin{cases} \dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ R(s) = \dfrac{1}{s} \end{cases}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

? $\zeta\omega_n = \dfrac{\zeta}{\sqrt{1-\zeta^2}}\,\omega_d$

$$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s + a)^2 + \omega^2}$$

$$\mathcal{L}[e^{-at}\sin\omega t] = \frac{s + a}{(s + a)^2 + \omega^2}$$ [3]

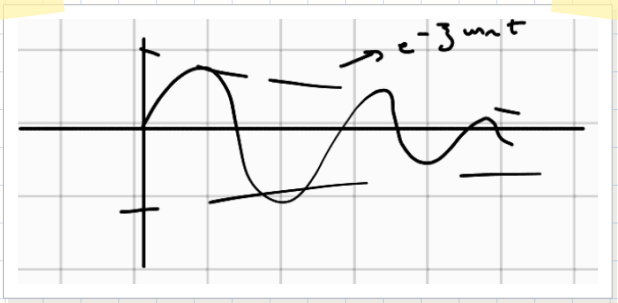
[3] Claude notu: muhtemelen cos ωt olmalı.

orig. p. 9

$$c(t) = \mathcal{L}^{-1}[C(s)] = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$



original drawing, p. 9

$$e(t) = r(t) - c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$