Kontrol Sistemleri

2nd/higher order systems, time-domain specs

Original notes: Control-Systems-lecture-7.pdf

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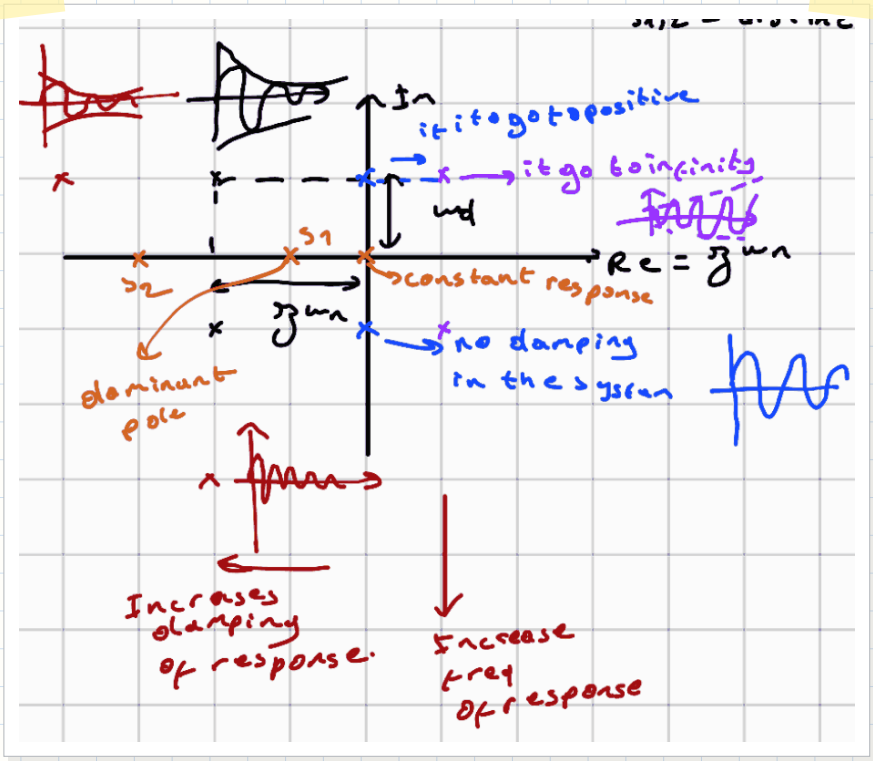
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orig. p. 1

In second order eq

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \begin{cases} \zeta < 1 \rightarrow s_{1,2} = \text{complex conjugates} \\ \zeta = 1 \rightarrow s_{1,2} = \text{repeated poles and real} \\ \zeta > 1 \rightarrow s_{1,2} = \text{distinct real roots} \end{cases}$$



original drawing, p. 1

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$
$$e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

Higher order Systems

$$\frac{X(s)}{U(s)} = \frac{\overbrace{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_m}^{\text{zeros of the system}}}{\underbrace{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}_{\text{poles of the system}}} \qquad m < n$$

$$X(s) = \text{''}\, U(s)$$

Step inputs are mostly used in analysis.

If the poles are real and distinct

$$\frac{X(s)}{U(s)} = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \cdots + \frac{r_n}{s+p_n} = \sum_{i=1}^{n}\frac{r_i}{s+p_i}$$

$$\overbrace{r_i}^{\text{residual}} = (s+p_i)\left.\frac{X(s)}{U(s)}\right|_{s=-p_i}$$

$$X(s) = U(s)\sum_{i=1}^{n}\frac{r_i}{s+p_i}$$

If $U(s)$ is unity (impulse)

$$\Rightarrow\ x(t) = \sum r_i e^{-p_i t}$$

orig. p. 2

If $U(s) = \frac{1}{s}$

$$X(s) = \frac{1}{s}\left(\frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}\right)$$
$$= \frac{r_0}{s} + \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \cdots + \frac{r_n}{s+p_n}$$

$$\Rightarrow\ x(t) = a_0 + \sum_{i=1}^{n} r_i e^{-p_i t}$$

if poles are not all real valued

for $U(s) = \frac{1}{s}$

$$X(s) = \frac{a_0}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_{n_k}) + c_k\,\omega_{d_k}}{s^2 + 2\zeta_k\omega_{n_k}s + \omega_{n_k}^2}$$

'q' poles are real valued

'2r' poles are complex valued

$$q + 2r = n$$

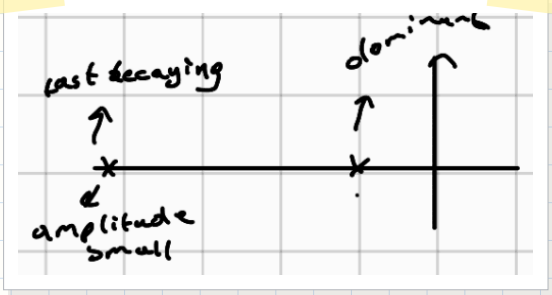
$$x(t) = a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_{n_k} t} \cos \omega_{d_k} t + \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_{n_k} t} \sin \omega_{d_k} t$$

$$\omega_{d_k} = \omega_{n_k} \sqrt{1 - \zeta_k^2}$$

if $z_1 = p_1 \Rightarrow r = 0$

zero and pole cancelation

if poles go most further to the jw axis the residues decrease



original drawing, p. 2

what we need

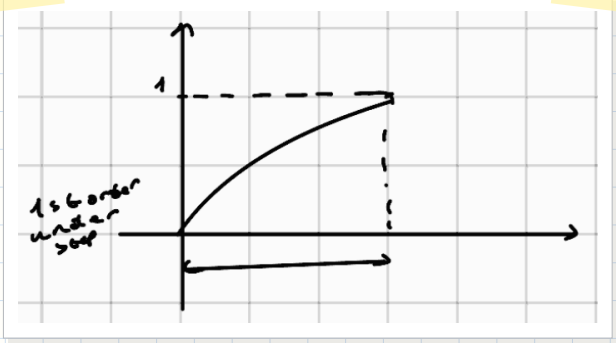
- Stability
- Accuracy (error)
- Quality of response

orig. p. 3

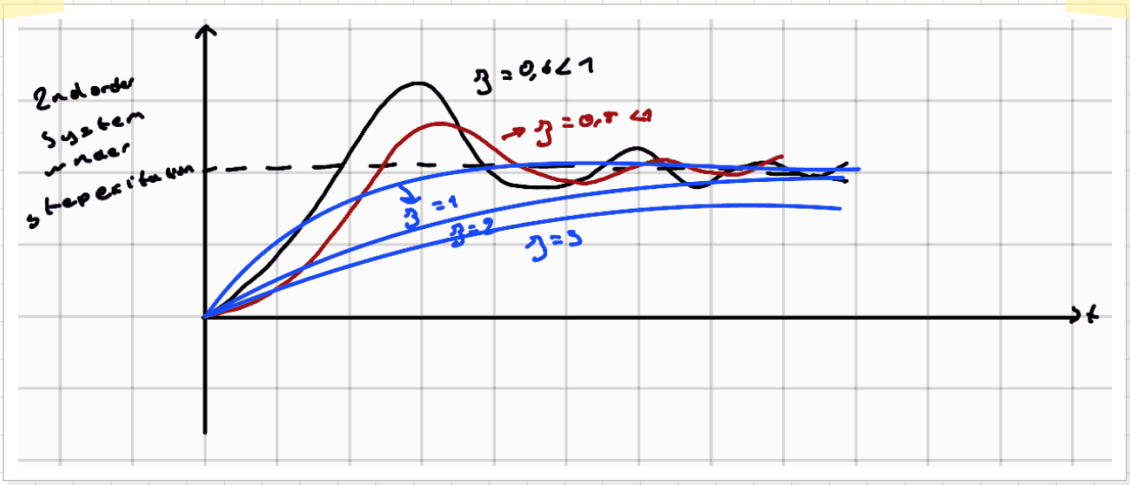
Quality of response

→ it focus on how we get to the steady state response

& later we'll be covering accuracy and stability of response as well.



original drawing, p. 3



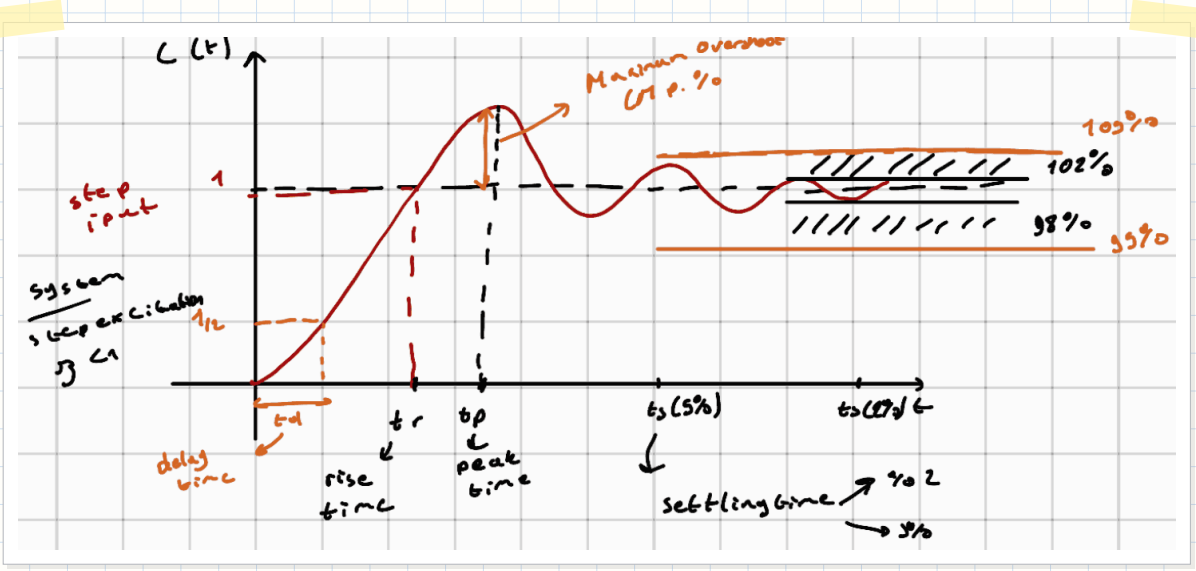
original drawing, p. 3

damping : Re : $\zeta\omega_n = \sigma$ (sigma)

Freq : Im : $\omega_n\sqrt{1-\zeta^2} = \omega_d$

$\zeta = 1$ critically damped case

settles down (gets to the final response) in the minimum amount of time



original drawing, p. 3

orig. p. 4

$\zeta < 1$: time needed to get from 0% to 100%

$\zeta > 1$: " " " " " 10% to 90%

rise time (t_r)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$1 = \underbrace{1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)}_{0}$$

$$\Rightarrow \cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

$e^{-\zeta\omega_n t_r}$ can not be zero in finite time

only when $t_r \to \infty \;\to\; e^{-\zeta\omega_n t_r} \to 0$

$$1 + \frac{\zeta}{\sqrt{1-\zeta^2}} \tan \omega_d t_r = 0$$

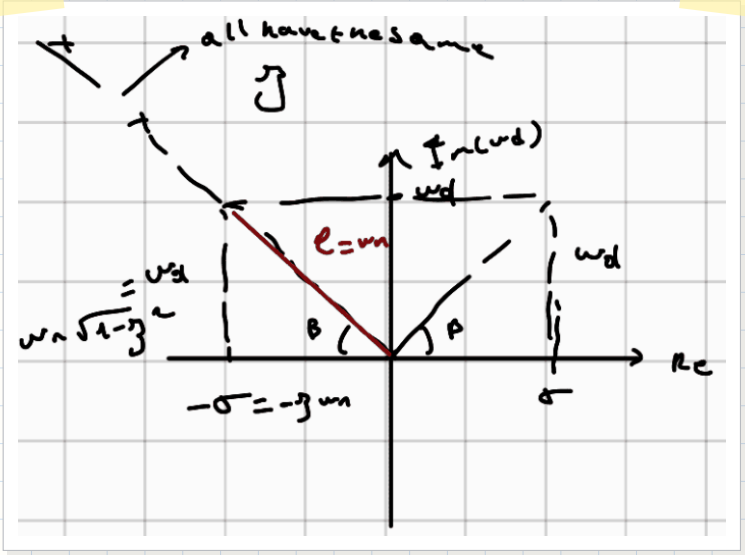
$$\tan \omega_d t_r = \frac{-\sqrt{1-\zeta^2}}{\zeta}$$

$$\Rightarrow \omega_d t_r = \tan^{-1}\left(\frac{-\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\Rightarrow t_r = \frac{1}{\omega_d} \tan^{-1}\left(\frac{-\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta}\right)$$

$$= \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{-\sigma}\right)$$

$$= \frac{\pi - \beta}{\omega_d} = t_r$$



original drawing, p. 4

$$l^2 = (\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)$$

$$l^2 = \omega_n^2$$

$$\cos\beta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

$$\beta = \cos^{-1}\zeta$$

peak time (t_p):

$$\frac{dc}{dt} = 0$$

$$\zeta\omega_n e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right)$$

$$- e^{-\zeta\omega_n t_p}\left(-\omega_d \sin\omega_d t_p + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t_p\right) = 0$$

orig. p. 5

$$\Rightarrow e^{-\zeta\omega_n t_p}\left(\zeta\omega_n\cos\omega_d t_p + \frac{\zeta^2\omega_n^2}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t_p + \omega_d\sin\omega_d t_p - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t_p\right) = 0$$

$$\left(\frac{\zeta^2\omega_n^2}{\omega_d} + \omega_d\right)\sin\omega_d t_p = 0$$

$$\Rightarrow \omega_d t_p = \pi, 2\pi, 3\pi, \ldots$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

$$c(t_p) = 1 - e^{-\zeta\omega_n\frac{\pi}{\omega_d}}\left(\overbrace{\cos\pi}^{-1} + \frac{\zeta}{\sqrt{1-\zeta^2}}\overbrace{\sin\pi}^{0}\right)$$

$$= 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

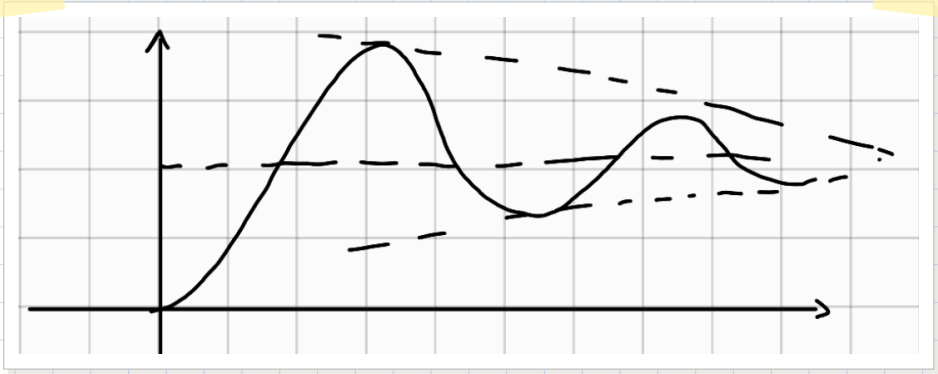
$$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$M_p = 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} - 1 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$= e^{-\frac{\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}} = e^{-\frac{\pi\sigma}{\omega_d}}$$

Settling time (t_s)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$



original drawing, p. 5

$$c(t) = 1 - a_1\underbrace{\sin(\omega_d t + \alpha)}_{-1<''<1}$$

$$= 1 \pm a_1$$

$$\sin(\omega_d t + \alpha) = \sin\omega_d t\cos\alpha + \cos\omega_d t\sin\alpha$$

orig. p. 6

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

$$\sin(\omega_d t + \alpha) = \sin\omega_d t \cos\alpha + \cos\omega_d t \sin\alpha$$

$$\begin{cases} \cos\alpha = \zeta \\ \sin\alpha = \sqrt{1-\zeta^2} \end{cases} \Rightarrow \tan\alpha = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

$$\text{bounding curves} : 1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} = \begin{cases} 1 \pm 0{,}05 & 5\% \\ 1 \pm 0{,}02 & 2\% \end{cases}$$

$$5\% \Rightarrow \frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0{,}05 \Rightarrow e^{-\zeta\omega_n t_s} = 0{,}05\sqrt{1-\zeta^2}$$

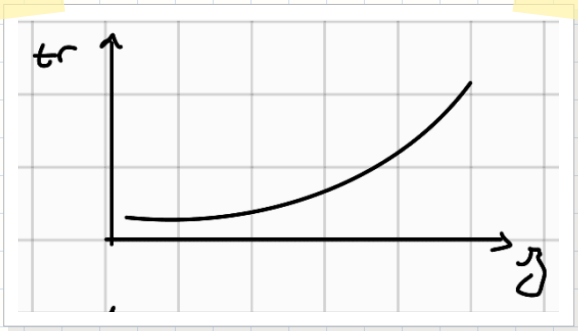
$$\Rightarrow -\zeta\omega_n t_s = \ln 0{,}05 + \ln\sqrt{1-\zeta^2}$$

$$\Rightarrow t_s = -\frac{\ln 0{,}05}{\zeta\omega_n} - \underbrace{\frac{\ln\sqrt{1-\zeta^2}}{\zeta\omega_n}}_{0} \quad \text{for small } \zeta \text{ values}$$

$$\Rightarrow t_s = \frac{3}{\zeta\omega_n} = 3T \quad \text{because} \quad \frac{1}{\zeta\omega_n}$$

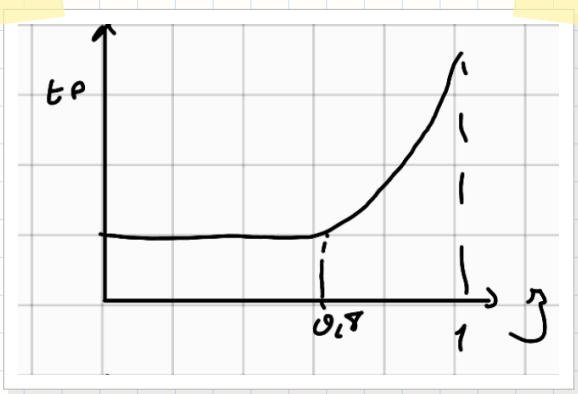
orig. p. 7

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{-\omega_d}{\sigma}\right)$$



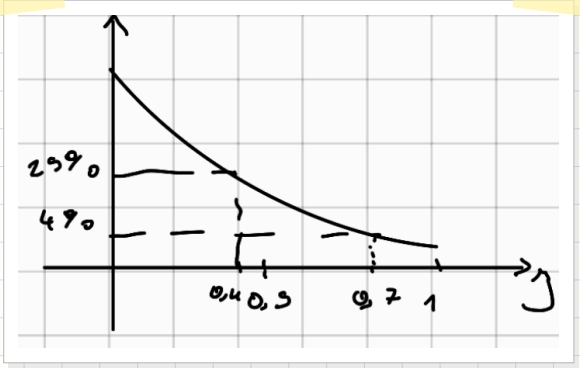
original drawing, p. 7

$$t_p = \frac{\pi}{\omega_d}$$



original drawing, p. 7

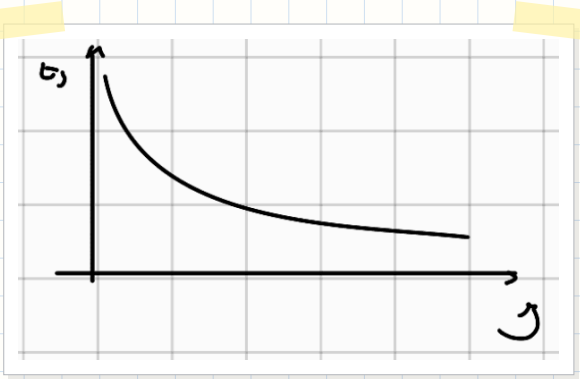
$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$



original drawing, p. 7

smaller

$$t_s = \frac{3}{\zeta\omega_n} \text{ or } \frac{4}{\zeta\omega_n}$$



original drawing, p. 7

M_p and t_r are contradictory !!

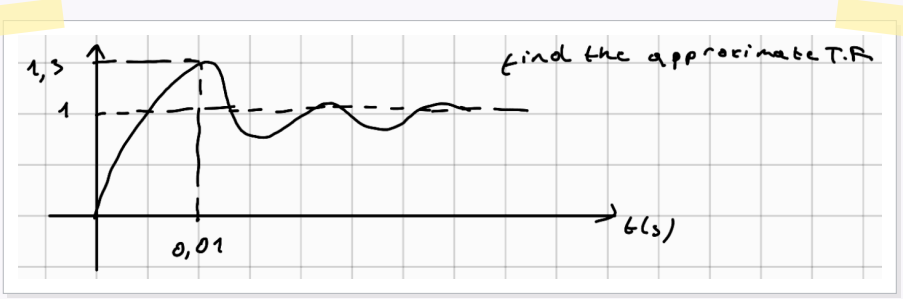
t_d = ? delay time

$$\frac{1}{2} = c(t)$$

Example

Step response of a 2nd order system is given

find the approximate T.F



original drawing, p. 7

orig. p. 8

$$t_p = 0{,}01$$

$$M_p = 0{,}3 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = 0{,}356$$

$$t_p = 0{,}01 = \frac{\pi}{\omega_d} \rightarrow \omega_d = 100\pi \Rightarrow \omega_n\sqrt{1-\zeta^2} = 100\pi$$

$$\Rightarrow \omega_n = 336$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{(336)^2}{s^2 + 2(0{,}356)(336)s + (336)^2}$$

Example

$\frac{C(s)}{R(s)} = \frac{k}{s^2 + as + k}$ for step input

$k, a = ?$

Maximum overshoot ≤ 10%

Settling time ≤ 1 sec (5%)

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \leq 0{,}1 \Rightarrow \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \leq \ln(0{,}1)$$

$$\Rightarrow \frac{\pi\zeta}{\sqrt{1-\zeta^2}} \geq \ln(0{,}1)^{1} \Rightarrow \zeta > 0{,}6$$

$$t_s = \frac{3}{\zeta\omega_n} \leq 1$$

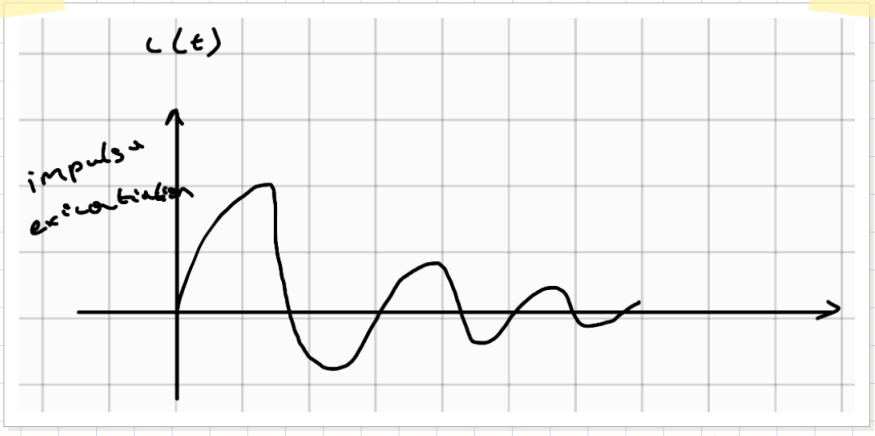
$$\zeta\omega_n \geq 3$$

$$\begin{cases} \zeta = 0{,}7 \\ \zeta\omega_n = 4 \end{cases} \Rightarrow \omega_n = \frac{4}{0{,}7}$$

$$\begin{cases} \omega_n^2 = k \\ 2\zeta\omega_n = a \end{cases}$$

orig. p. 9

SISO Tool Matlab



original drawing, p. 9

[1] Claude notu: muhtemelen ≥ −ln(0,1) olmalı.

Example

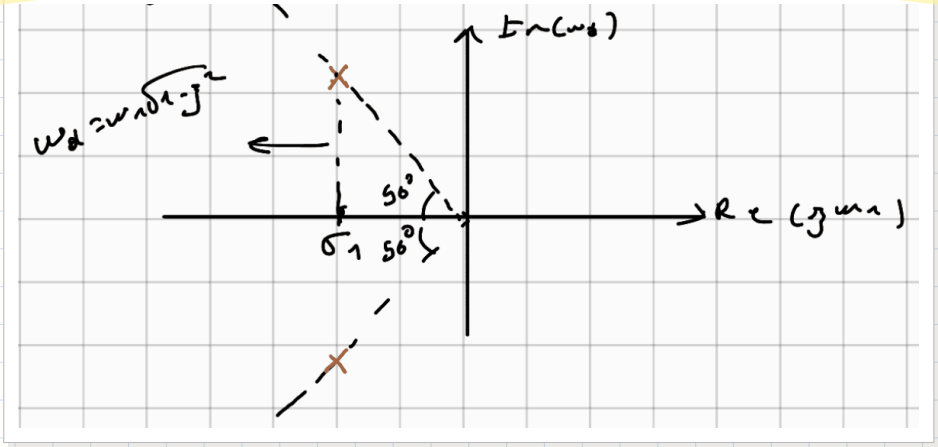
Find the specific location of poles on the s plane

a) $M_p = 12\%$, $t_s = 0{,}6$ sec ? stable

b) $M_p = 10\%$, $t_s = 3$ sec

$$M_p = 0{,}12 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = 0{,}559$$

$$5\% \quad t_s = \frac{3}{\zeta\omega_n} = 0{,}6 \Rightarrow \omega_n = 8{,}94 \text{ rad/s}$$



original drawing, p. 9

$$\zeta\omega_n = 0{,}559 \cdot 8{,}94 = \sigma_1$$

$$\beta = \cos^{-1}\zeta = \cos^{-1} 0{,}559 = 56°$$

$$p_1 = -\sigma_1 + j\overbrace{\omega_n \sin\beta}^{\omega_n\sqrt{1-\zeta^2}}$$

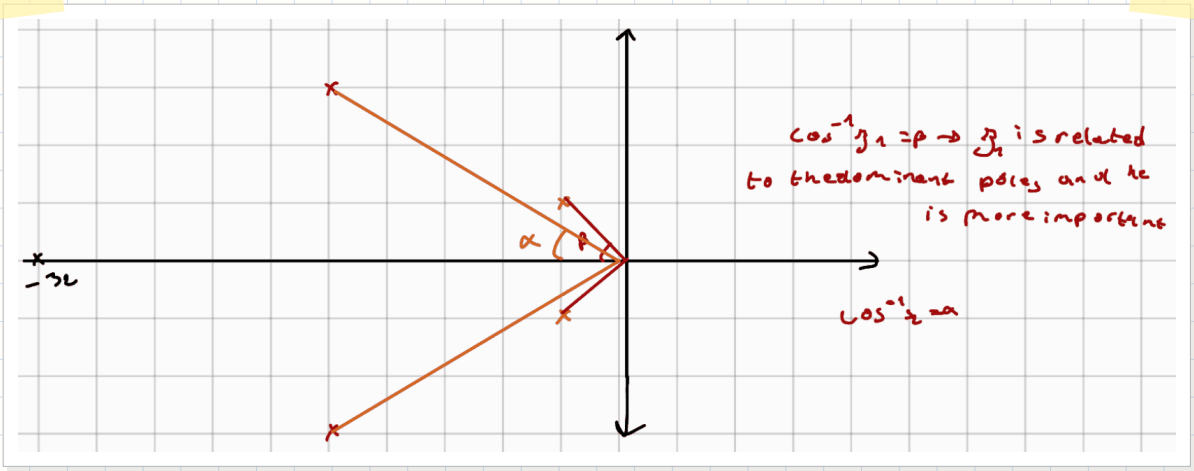
$$p_2 = -\sigma_1 - j\omega_n \sin\beta$$

orig. p. 10

Find ζ for the given C.L.T.F.

$$G(s) = \frac{1638\,(s^2 + 2{,}6\,s + 1{,}63)}{\underbrace{(s + 3{,}24)}_{\text{real pole}}\ \underbrace{(s + 0{,}855 + j0{,}672)(s + 0{,}855 - j0{,}672)}_{\text{c.c. pole}}\ \underbrace{(s + 5{,}24 + j6{,}56)(s + 5{,}24 - j6{,}56)}_{\substack{\text{c.c.} \\ \text{with negative real part}}}}$$

with negative real part



original drawing, p. 10

The system is combine of 2 second degree systems

by looking to the dominant poles 2 can be found

$$\tan\beta = \frac{\omega_d}{\sigma} = \frac{0{,}672}{0{,}855} \rightarrow \beta = 38{,}16^\circ \Rightarrow \zeta_1 = 0{,}78$$

$$\tan\alpha = \frac{6{,}56}{5{,}24} \rightarrow \alpha = 51{,}38^\circ \Rightarrow \zeta_2 = 0{,}62$$