Kontrol Sistemleri

Accuracy of Systems: Steady-State Error (Ders 9)

Original notes: Control-Systems-lecture-9.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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orig. p. 1

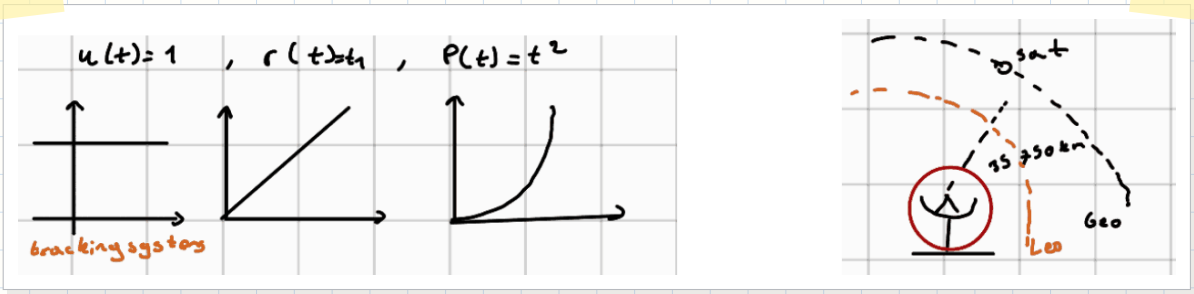
Ders 9

Accuracy of Systems

The ability of the system follow the given instruction can be used as a measure of classification.

* Common input: step, ramp, parabolic

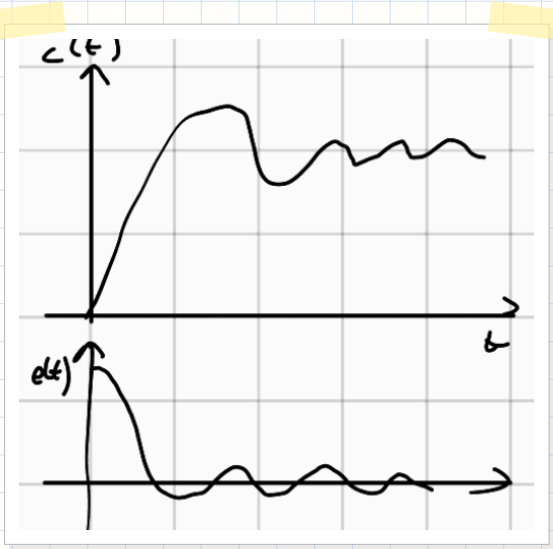
$$u(t) = 1 \quad , \quad r(t) = t \quad , \quad P(t) = t^2$$



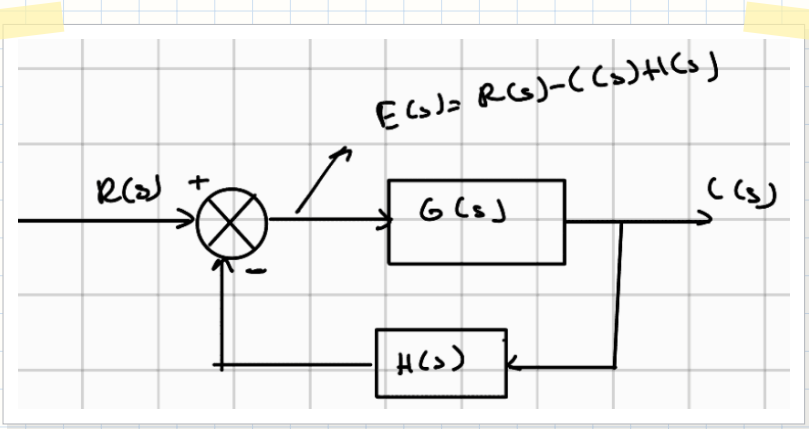
original drawing, p. 1

* Comparison of accuracy is possible by using the steady-state error magnitude!!

if $\zeta < 1$ in a second o.s



original drawing, p. 1



original drawing, p. 1

$$E(s) = R(s) - C(s)H(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$E(s) = R(s) - G(s)H(s) \Rightarrow \frac{E(s)}{R(s)} = 1 - \frac{G(s)H(s)}{R(s)}$$

$$\Rightarrow \frac{E(s)}{R(s)} = 1 - \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{1}{1 + G(s)H(s)}$$

$$e(\infty) = e_{ss} = \lim_{t \to \infty} e(t) \overset{\text{Final value theorem}}{=} \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

orig. p. 2

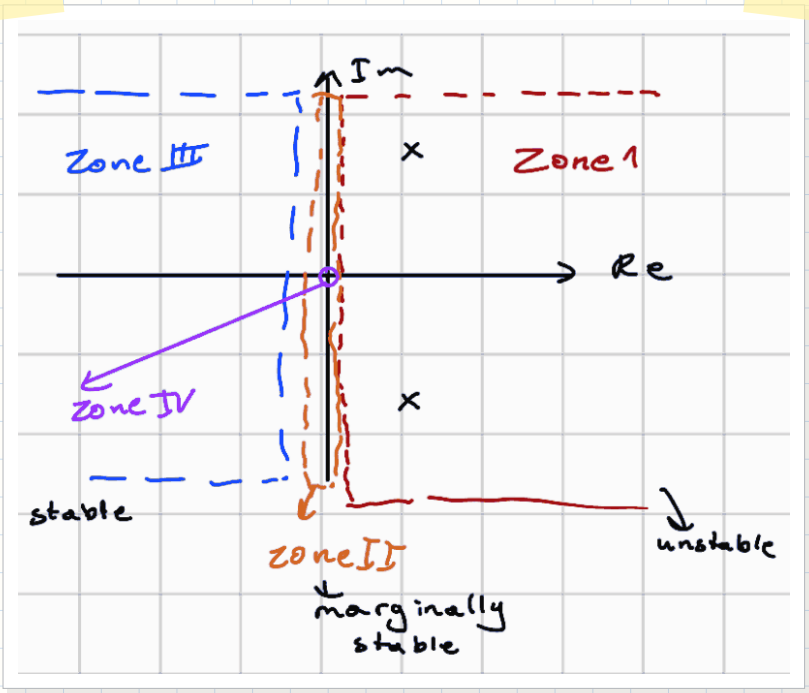
Validity of final value theorem

If the poles are in zone 1 so it can't have final value

Only works with Zone III and zone IV

Stability → CLP ✓

OLP



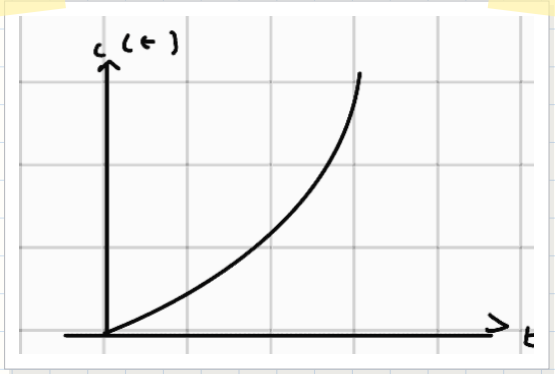
original drawing, p. 2

Zone I :

$$G(s) = \text{T.F} = \frac{1}{s-2}$$

$$\text{FVT} = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s}{s-2} = 0 \quad \times \text{ wrong}$$

but



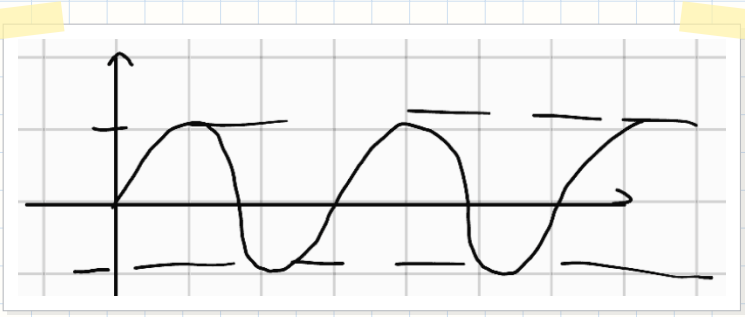
original drawing, p. 2

For Zone II

$$\text{T.F} = G(s) = \frac{1}{s^2+2}$$

$$s_{1,2} = \pm j\sqrt{2}$$

$$\text{FVT}: \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s}{s^2+2} = 0 \quad \times$$

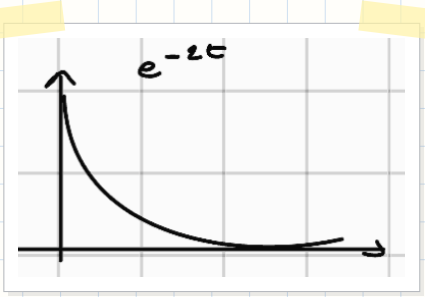


original drawing, p. 2

Zone III

$$T.F = G(s) = \frac{1}{s+2} \quad \Rightarrow s = -2$$

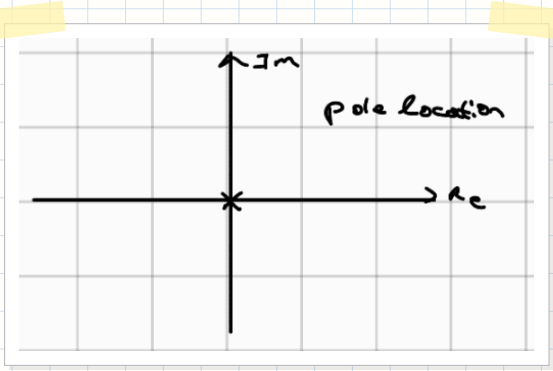
$$FVT: \lim_{s \to 0} \frac{s}{s+2} = 0$$



original drawing, p. 2

orig. p. 3

Zone IV :



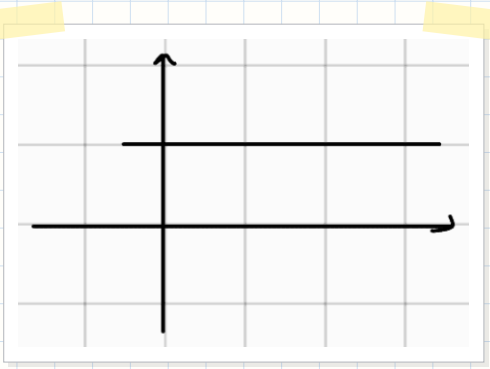
original drawing, p. 3

$$TF = \frac{1}{s}$$

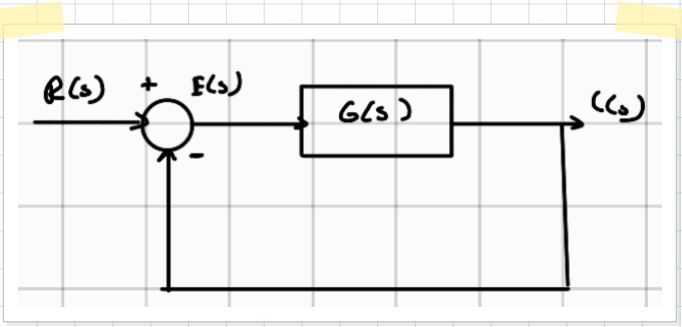
$$FVT = \lim_{s \to 0} s \frac{1}{s} = 1$$

$$\lambda = 0$$

$$e^{\lambda t} = e^{0t} = 1$$



original drawing, p. 3

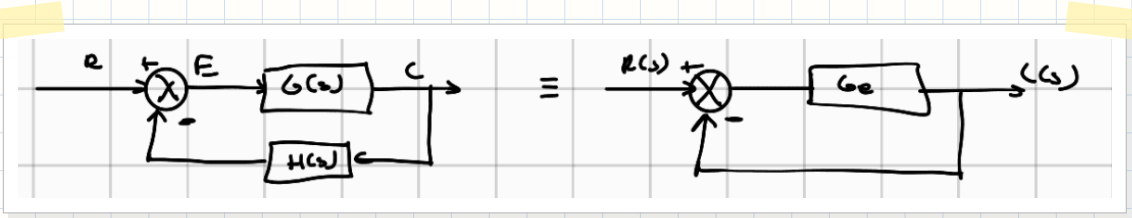


original drawing, p. 3

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$G(s)$: open loop TF for unity feedback

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$



original drawing, p. 3

$$G(s) = \frac{k\,(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

$P_{1,\ldots,N} = 0 \leftarrow$ N multiplicity $\rightarrow$ integrator

$p = \frac{-1}{T_1}, \frac{-1}{T_2}, \ldots$

if N=0 $\rightarrow$ type zero

if N=1 $\rightarrow$ Type one

⋮

In most cases the most integrator is 3

orig. p. 4

Accuracy of systems

case 1) step input: $R(s) = \frac{1}{s}$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+k_p}$$

? constant

$$k_p = \lim_{s\to 0} G(s) \qquad \text{(static position error)} \qquad \text{DC gain}$$

$$\text{Type 0:} \quad G(s) = \frac{k(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

$$N = 0 \Rightarrow k_p = \lim_{s\to 0} G(s) = k$$

$$N = 1 \Rightarrow k_p = \lim_{s\to 0} G(s) = \infty$$

$$N = 2 \Rightarrow k_p = \lim_{s\to 0} = \infty$$

$$N \geq 1 \Rightarrow k_p = \infty$$

case 2) input is ramp

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s\to 0} \frac{s\frac{1}{s^2}}{1+G(s)} = \lim_{s\to 0} \frac{1}{s(1+G(s))}$$

$$= \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{k_v}$$

k_v = static velocity error = $\lim_{s\to 0} sG(s)$

$$\text{Type 0:} \quad k_v = \lim_{s\to 0} sG(s)$$

$$= \lim_{s\to 0} \frac{s\,k(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = 0$$

$$\text{Type 1:} \quad k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s\,k(T_a s+1)\cdots}{s(T_1 s+1)} = k$$

$$\text{Type 2:} \quad k_v = \lim_{s\to 0} \frac{s\,k(T_a s+1)\cdots}{s^2(T_1 s+1)\cdots} = \infty$$

case 3)

$$R(s) = \frac{1}{s^3}$$

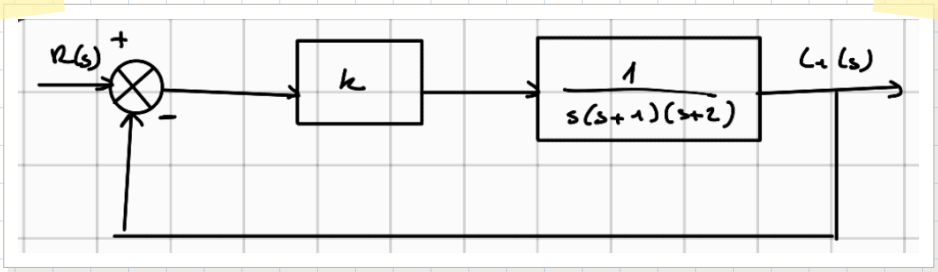
$$e_{ss} = \lim_{s\to 0} \frac{s\frac{1}{s^3}}{1+G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)} = \frac{1}{k_a} \rightarrow \text{static acceleration error}$$

$$N = 0: \quad k_a = \lim_{s \to 0} \frac{s^2 k(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1) \cdots} = 0$$

orig. p. 5

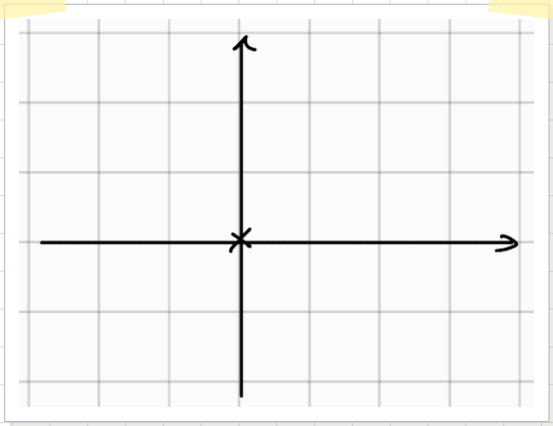
	input		
Type	step	ramp	parabolic
0	$\frac{1}{1+k}$	∞	∞
1	0	$\frac{1}{k}$	∞
2	0	0	$\frac{1}{k}$
3	0	0	0

Example :



original drawing, p. 5

$$\text{OL T.F} = \frac{k}{s(s+1)(s+2)} \qquad \text{Type 1 system}$$



original drawing, p. 5

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{s \cdot \frac{1}{s^2}}{1 + \frac{k}{s(s+1)(s+2)}} = \lim_{s\to 0} \frac{s(s+1)(s+2)}{s\left[s(s+1)(s+2)+k\right]}$$

$$= \frac{2}{k} = 0{,}01 \Rightarrow k = 200 \checkmark$$

$\uparrow$ for $e_{ss} = 0{,}01$ design

But we have to check stability

$$\frac{\frac{k}{s(s+1)(s+2)}}{1 + \frac{k}{s(s+1)(s+2)}} = \frac{k}{s(s+1)(s+2)+k}$$

orig. p. 6

Stability

$k \to p$

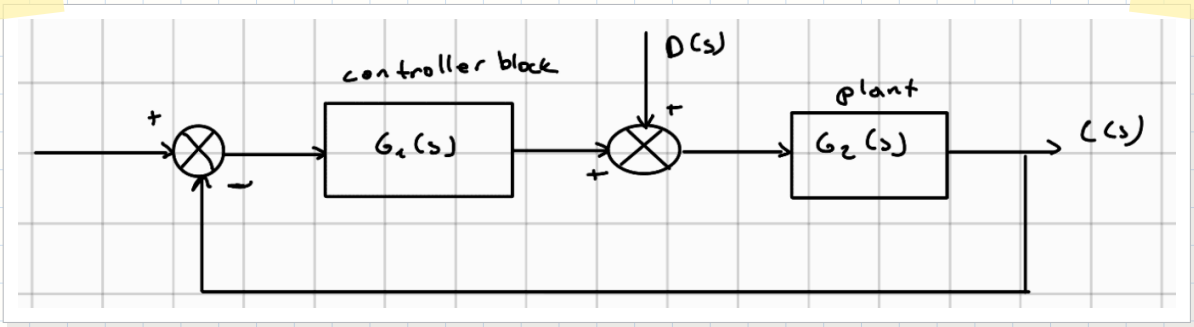
$$s(s+1)(s+2) + k = 0$$
$$s(s^2 + 3s + 2) + k = 0$$
$$s^3 + 3s^2 + 2s + k = 0$$

s^3	1	2	
s^2	3	k	$\Rightarrow 6 - k \geq 0 \Rightarrow k \leq 6$
s^1	$\frac{6-k}{3}$	0	$k \geq 0$
s^0	k		$0 \leq k \leq 6$

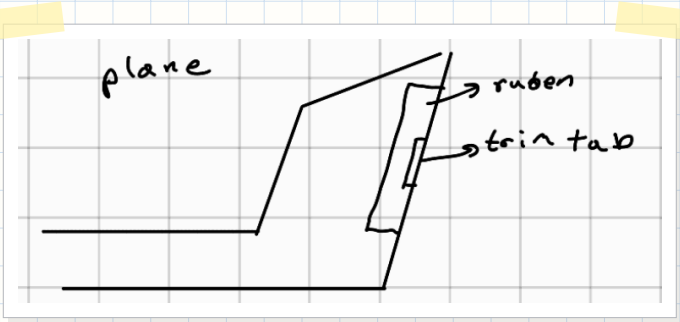
$k = 6$ is the max possible k here !!!

so max $e_{ss} = 0{,}33$ because the system have to be stable because final value theorem is not working with unstable systems.

Steady state error for disturbance



original drawing, p. 6



original drawing, p. 6

$$C(s) = \left[E(s)G_1(s) + D(s)\right] G_2(s)$$
$$E(s) = R(s) - C(s) \Rightarrow C(s) = R(s) - E(s)$$
$$E(s) = \frac{R(s)}{1 + G_1(s)G_2(s)} - \frac{G_2(s)D(s)}{1 + G_1(s)G_2(s)}$$

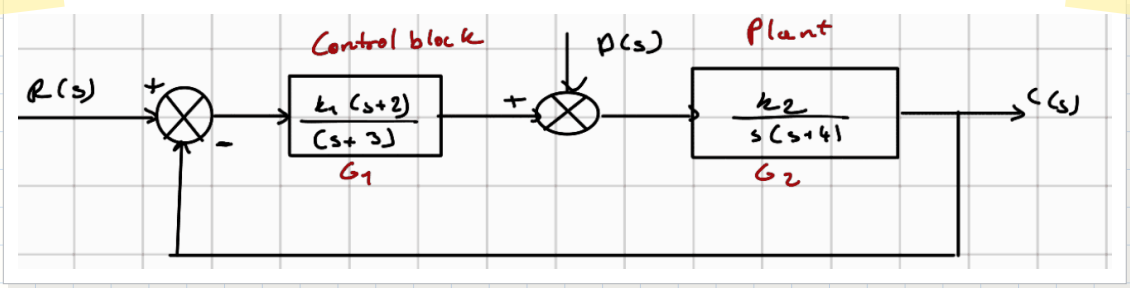
$$e_{ss} = \lim_{s\to 0} sE(s)$$
$$= \underbrace{\lim_{s\to 0} \frac{sR(s)}{1 + G_1G_2}}_{e_R(\infty)} - \underbrace{\lim_{s\to 0} \frac{sG_2D}{1 + G_1G_2}}_{e_D(\infty)}$$

orig. p. 7

because it is negative we can decrease error with it

Example

Design k_1 and k_2 so that
1 - e_{ss} due to unit step disturbance : $-0{,}00001$
2 - e_{ss} " " ramp input : $0{,}002$



original drawing, p. 7

$$\frac{k_1(s + 3 - 1)}{s + 3} = \frac{k_1(s + 3)}{s + 3} - \frac{k_1}{s + 3} = k_1 - \frac{k_1}{s + 3}$$

$$e_{ss}^D = -\lim_{s\to 0} \frac{s\,G_2(s)D(s)}{1 + G_1(s)G_2(s)} = -\lim_{s\to 0} \frac{G_2(s)}{1 + G_1(s)G_2(s)} = \frac{-3}{2k_1}$$

$$-0{,}00001 = \frac{-3}{2k_1} \Rightarrow k_1 = 150\,000$$

$$e_{ss}^{R} = \lim_{s\to 0} \frac{sR(s)}{1 + G_1(s)G_2(s)} = \lim_{s\to 0} \frac{1}{s\,(1 + G_1(s)G_2(s))}$$

$$= \frac{6}{k_1 k_2} = 0{,}002 \Rightarrow k_2 = 0{,}02$$

$$\text{Total } e_{ss} = \frac{6}{k_1 k_2} - \frac{3}{2k_1}$$

$$= 0{,}02^{1} - 0{,}00001$$

this system order is 3 by finding characteristic equation.

Sensitivity

The degree to which changes in the system parameters affect system T.F. and hence performance *(error, stability, ...)* of the system

orig. p. 8

Example

$$F = \frac{k}{k + a} \quad \text{if } k = 10$$

$$F = \frac{10}{10 + a} \qquad \begin{cases} a = 100 \rightarrow F = \dfrac{10}{110} = 0{,}09 \\ \quad (+200\%) \\ a = 300 \rightarrow F = \dfrac{10}{310} = 0{,}032 \end{cases} \qquad \curvearrowright -65\%$$

$$\frac{\Delta F}{F} \times 100 = \frac{0{,}032 - 0{,}091}{0{,}091} \times 100 = -65\%$$

$$\frac{\Delta a}{a} \times 100 = \frac{300 - 100}{100} \times 100 = 200\%$$

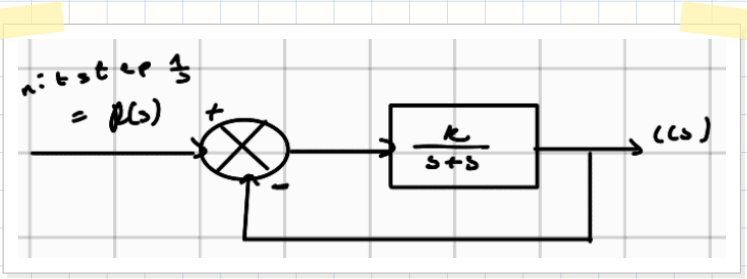
(Reduced sensitivity)
This means it's low sensitive to a
Average sensitivity : $\frac{65\%}{200\%} < 1$ (Reduced sensitivity)

$$S_{F:a} = \lim_{\Delta p\to 0} \frac{\text{fractional change in function}(F)}{\text{" " " parameter}(p)}$$

$$= \lim_{\Delta p\to 0} \frac{\Delta F/F}{\Delta p/p} = \lim_{\Delta p\to 0} \frac{p}{F}\frac{\Delta F}{\Delta p}$$

$$S_{F:a} = \frac{p}{F}\frac{\partial F}{\partial p}$$

[1] Claude notu: muhtemelen 0,002 – 0,00001 olmalı.



original drawing, p. 8

Check the sensitivity of ss. error w.r.t. parameter 'k'

$$e(\infty) = e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{1}{1+G(s)}$$

$$= \frac{1}{1+\lim_{s\to 0} \frac{k}{s+5}} = \frac{5}{5+k}$$

$$S_{e:k} = \frac{k}{e_{ss}} \cdot \frac{\partial e_{ss}}{\partial k} = \left(\frac{k}{\frac{5}{5+k}}\right)\left(-\frac{5}{(5+k)^2}\right)$$

$$= \frac{k(5+k)}{5}\left(-\frac{5}{(5+k)^2}\right) = -\frac{k}{5+k}$$

for $k = 10 \Rightarrow S_{e:k} = \frac{-10}{5+10} = \frac{-10}{15} = \frac{-2}{3}$

if 'k' is changed to 12

$$\frac{12-10}{10} \leftarrow \frac{\Delta k}{k} \times 100 = 20\%$$

$\delta \rightarrow$ delta also

$$\underbrace{S_{e:k}}_{-\frac{2}{3}} \cdot \underbrace{\frac{\Delta k}{k}}_{\frac{1}{5}} = \frac{\delta e_{ss}}{e_{ss}} = -\frac{2}{3} \times \frac{1}{5} = \frac{-2}{15} = -13\%$$

orig. p. 9

$$\delta e_{ss} = -0{,}13 \times e_{ss}$$

$$= -0{,}13 \times \frac{1}{3} = -0{,}04 \quad \text{the amount of change}$$