Nümerik Analiz

8. Hafta – Nümerik İntegralleme

Orijinal not: Numerik-8.hafta.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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orj. s. 1

Nümerik integralleme

Belirli integraller

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \alpha_k f(x_k) \qquad \text{"quadrature"}$$

$$x_0, x_1, x_2, \ldots, x_n \in [a,b]$$

Lagrange IP $\qquad P_n(x) = \sum_{k=0}^{n} \underbrace{L_{n,k}(x)}_{\prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}} f(x_k)$

$f \in C^{n+1}[a,b]$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \qquad \xi = \xi(x) \in (a,b)$$

$$\int_a^b f(x)\,dx = \underbrace{\int_a^b \sum_{k=0}^{n} L_{n,k}(x) f(x_k)\,dx}_{\text{Tahmin}} + \underbrace{\int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)\,dx}_{\text{Hata}}$$

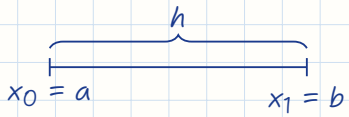
$$= \underbrace{\sum_{k=0}^{n} \alpha_k f(x_k)}_{\text{Tahmin}} + \underbrace{E(f)}_{\text{Hata}}$$

$$\alpha_k = \int_a^b L_{n,k}(x)\,dx$$

Trapezoid (yamuk) kuralı

$$f(x) \cong P_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$E_1(x) = \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1) \qquad \xi \in (x_0, x_1)$$



$$\int_a^b f(x)\,dx = \int_a^b P_1(x)\,dx + \int_a^b E_1(x)\,dx$$

Teorem (İntegral için ortalama (ara) değer teoremi)

$f \in C[a,b]$ ve $g(x)$ $a \le x \le b$ integrallenebilir ve işaretini korur öyle bir $c \in (a,b)$ var ki,

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

$$\int_{x_0}^{x_1} f''(\xi(x)) \underbrace{(x-x_0)(x-x_1)}_{g(x)\text{ (işaret değiştirmiyor)}} dx$$

$$= f''(\xi_1)\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx\Big|_{h=x_1-x_0} = -\frac{h^3}{6} f''(\xi_1)$$

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(Üstte karalanmış: $-\frac{h^3}{6} f''(\xi_1)$)

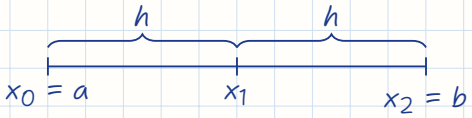
$$\int_a^b f(x)\,dx = \int_a^b P_1(x)\,dx + \int_a^b E_1(x)\,dx$$

$$= \left[\frac{(x-x_1)^2}{2(x_0-x_1)} f(x_0) + \frac{(x-x_0)^2}{2(x_1-x_0)} f(x_1)\right]_{x=x_0}^{x=x_1}$$

$$+ -\frac{h^3}{12} f''(\xi_1)$$

$$= \frac{h}{2}\left[f(x_0)+f(x_1)\right] - \frac{h^3}{12} f''(\xi_1)$$

Simpson kuralı



$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx + \int_{x_0}^{x_1} E_2(x)\,dx$$

(x_1 civarından Taylor:)

$$f(x) = f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2!}(x-x_1)^2 + \frac{f'''(x_1)}{3!}(x-x_1)^3$$

$$+ \frac{f^{(4)}(\xi)}{4!}(x-x_1)^4$$

$$\int_a^b f(x)\,dx = \Big[f(x_1)(x-x_1) + \frac{f'(x_1)}{2}(x-x_1)^2 + \frac{f''(x_1)}{6}(x-x_1)^3 + \frac{f'''(x_1)}{24}(x-x_1)^4\Big]_{x_0}^{x=x_2} + \underbrace{\int_{x_0}^{x_2} \frac{f^{(4)}(\xi(x))}{24}(x-x_1)^4\,dx}_{\frac{f^{(4)}(\xi_1)}{120}\,2h^5}$$

$$f'(x_1) = \frac{1}{h^2}\Big[f(x_0) - 2f(x_1) + f(x_2)\Big] - \frac{h^2}{12}f^{(4)}(\xi_2)$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx + \int_{x_0}^{x_1} E_2(x)\,dx = 2h\,f(x_1) + \dots$$

$$\swarrow \frac{h^3}{3}\left[\frac{1}{h^2}\Big[f(x_0) - 2f(x_1) + f(x_2)\Big] - \frac{h^2}{12}f^{(4)}(\xi_2)\right] + \frac{f^{(4)}(\xi_1)\,h^5}{60}$$

orj. s. 3

$$= \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] - \frac{h^5}{12}\left[\frac{1}{3}f^{(4)}(\xi_2) - \frac{1}{5}f^{(4)}(\xi_1)\right]$$

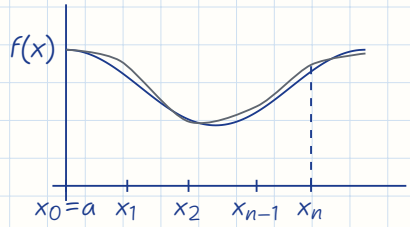
$$\int_a^b f(x)\,dx = \underbrace{\frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big]}_{\text{Tahmin}} - \underbrace{\frac{h^5}{90}f^{(4)}(\xi)}_{\text{Hata}}$$

Kapalı Newton-Cotes formülü (Genelleme)

$[a,b]$ kapalı aralık

$x_0 = a,\ x_n = b$

$x_k = x_0 + kh \qquad h = \dfrac{b-a}{n} \qquad k = 0, 1, \dots, n$



$$P_n(x) = \sum_{k=1}^{n} f(x_k) \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{k=0}^{n}(x-x_k)$$

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx = \sum_{k=0}^{\infty}{}^{1}\,\alpha_k f(x_k)$$

$$\alpha_k = \int_a^b \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}\,dx$$

[1] Claude notu: muhtemelen toplamın üst sınırı n olmalı.

Teorem

$\sum_{k=0}^{n} \alpha_k f(x_k)$ $(n+1)$ adet noktadan kapalı NC formülü olsun $n=1 \to$ Trapezoid, $n=2 \to$ Simpson

$x_0 = a$, $x_n = b$ ve $h = \dfrac{b-a}{n}$

Böylece $\xi \in (a,b)$ sayısı, n çift ve $f \in C^{n+2}[a,b]$ olacak biçimde

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} \alpha_k f(x_k) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2 (t-1)\cdots(t-n)\,dt$$

n tek ve $f \in C^{n+1}[a,b]$

" " $+ \dfrac{h^{n+1} f^{(n+1)}(\xi)}{(n+1)!} \displaystyle\int_0^n t(t-1)\cdots(t-n)\,dt$

formülleri ?sağlanır.

orj. s. 4

Açık Newton-Cotes formülü (Genelleme)

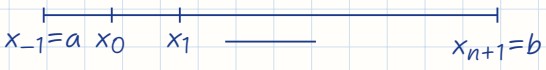
(a,b) açık aralık

$x_{-1} = a$, $x_{n+1} = b$

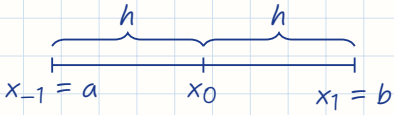
$x_k = x_0 + kh$ $\quad h = \dfrac{b-a}{n+2}$

$k = 0, 1, \ldots, n$

- $x_0, x_1, \ldots, x_n \in (a,b)$



$n = 0$ durumunda formülü belirleyelim $h = \dfrac{b-a}{2}$ $\quad x_0 = a + h$



$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2!}(x-x_0)^2$$

$$\int_a^b f(x)\,dx = \int_{x_{-1}}^{x_1} f(x_0)\,dx + \int_{x_{-1}}^{x_1} f'(x_0)(x-x_0)\,dx + \frac{1}{2}\int_{x_{-1}}^{x_1} f''(\xi(x))(x-x_0)^2\,dx$$

$$= f(x_0)(x_1 - x_{-1}) + f'(x_0)\frac{(x-x_0)^2}{2}\Big|_{x_{-1}}^{x_1} + \frac{1}{2} f''(\xi_1)\int_{x_{-1}}^{x_1} (x-x_0)^2\,dx$$

$$= 2h f(x_0) + \frac{h^3}{3} f''(\xi_1)$$

(Son satırın başında karalanmış bir ifade var.)

orj. s. 5

Uygulama

Örnek

1) $\int_{0,5}^{1} x^4\,dx = \frac{31}{160} = \frac{31}{160} = 0{,}19375$ 2) $\int_{e}^{e+1} \frac{1}{x\ln x}\,dx \cong 0{,}2725$

İntegrallerin değerini Trapezoidal ve Simpson formülleri kullanarak tahmini hesaplayınız

Çözüm: 1) $f(x) = x^4$, $\underbrace{0{,}5}_{a} \le x \le \underbrace{1}_{b}$ $x_0 = 0{,}5,\ x_1 = 1,\ h = b-a = 1-0{,}5 = 0{,}5$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0)+f(x_1)\right]$$

$$\int_{0,5}^{1} x^4\,dx = \frac{0{,}5}{2}\left[(0{,}5)^4 + 1^4\right] = \frac{1}{4}\left[\frac{1}{16}+1\right] = \frac{17}{64} \approx 0{,}26$$

Simpson

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] \qquad x_0 = 0{,}5 \quad x_1 = x_0 + h \quad x_2 = 1 \quad h = \tfrac{b-a}{2}$$

$$h = \frac{1-0{,}5}{2} = \frac{1}{4} = 0{,}25 \qquad \boxed{x_1 = 0{,}5 + 0{,}25 = 0{,}75}$$

$$\frac{1/4}{3}\left[\left(\frac{1}{2}\right)^4 + 4\left(\frac{3}{4}\right)^4 + 1^4\right]$$

$$= \frac{1}{12}\left[\frac{1}{16} + \frac{81}{64} + 1\right] = \frac{149}{12\cdot 64} = 0{,}1940$$

2) $x_0 = e \quad x_1 = x_0 + h \quad x_2 = e+1 \quad h = \frac{b-a}{2}$

$$h = \frac{e+1-e}{2} = \frac{1}{2}$$

$$x_1 = e + \frac{1}{2} = e + 0{,}5$$

$$\frac{1}{6}\left[\frac{1}{e\ln e} + 4\,\frac{1}{\left(\frac{1}{2}+e\right)\ln\left(\frac{1}{2}+e\right)} + \frac{1}{(1+e)\ln(1+e)}\right] \cong 0{,}2727$$

orj. s. 6

Çözüm: $x_0 = e,\ x_1 = 1+e \qquad h = b-a = 1$

$$\int_e^{e+1} \frac{1}{x\ln x}\,dx \cong \frac{1}{2}\left[\frac{1}{e\ln e} + \frac{1}{(1+e)\ln(1+e)}\right] \cong 0{,}2863$$

Örnek 3

Ara nokta formülünü kullanarak

$$\int_0^{\pi/4} \sin x\,dx = 1 - \frac{1}{\sqrt{2}} \approx 0{,}2929$$

bu integrali tahmini hesaplayınız

Çözüm:

$$\int_a^b f(x)\,dx \cong 2h\,f(x_0), \qquad h = \frac{b-a}{2}$$

$$x_0 = a + h$$

$a = 0$
$b = \pi/4$
$h = \dfrac{\pi/4 - 0}{2} = \pi/8$
$x_0 = 0 + \pi/8 = \pi/8$

$$\int_0^{\pi/4} \sin x\,dx = 2\,\frac{\pi}{8}\cdot \sin(\pi/8)$$

$$= \frac{3{,}14}{4}\,\sin\left(\frac{180}{8}\right) \approx 0{,}3005$$

orj. s. 7

Örnek 4

$\int_{-0{,}5}^{0} x\ln(1+x)\,dx$ Trapezoidal, Simpson ve ara formülleri kullanarak hesaplayın

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] - \frac{h^3}{12}\,f''(\xi) \qquad \text{Tra...}$$

(Sayfanın geri kalanı boş.)

orj. s. 8

(Bu sayfa tahtanın fotoğrafı; 3. sorunun tahtadaki çözümü.)

Örnek 3

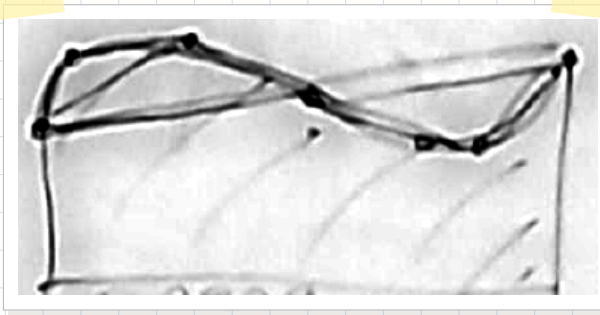
Ara nokta formülü kullanarak

$$\int_0^{\pi/4} \sin x\,dx = 1 - \frac{1}{\sqrt{2}} \approx 0{,}2929$$

bu integrali tahmini hesaplayın

$$\boxed{\int_a^b f(x)\,dx \cong 2h\,f(x_0)}$$

Ara nokta $h = \dfrac{b-a}{2}$
$x_0 = a + h$



orijinal çizim, s. 8

$a = 0$
$b = \pi/4$
$h = \dfrac{\pi/4 - 0}{2} = \pi/8$
$x_0 = 0 + \pi/8 = \pi/8$

$$\int_0^{\pi/4} \sin x\,dx = \underbrace{2\,\frac{\pi}{8}}_{\pi/4} \sin(\pi/8)$$
$$= \frac{3{,}14}{4} \sin\left(\frac{180^\circ}{8}\right)$$
$$\boxed{\approx 0{,}3005}$$

orj. s. 9 (Bu sayfa da tahtanın fotoğrafı; 4. soru.)

Örnek 4

$\int_{-0{,}5}^{0} x\ln(1+x)\,dx$ Trapezoidal, Simpson ve Ara nokta formülleri kullanarak hesaplayın

$$\int_a^b f(x)\,dx = \underbrace{\frac{h}{2}\left[f(x_0) + f(x_1)\right]}_{\text{tahmini değer}} - \underbrace{\frac{h^3}{12} f''(\xi)}_{\text{hata}}$$ Trapezoidal ✓

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \underbrace{\frac{h^5}{90} f^{(4)}(\xi)}_{\text{hata}}$$ Simpson ✓