Sayısal Haberleşme

Örnekleme teoremi ve örnekleme türleri

Orijinal not: Ders-4.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1



orijinal çizim, s. 1

Dirac katarı

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \overset{F}{\longleftrightarrow} X_\delta(f) = \overbrace{\frac{1}{T_s}}^{=f_s} \sum_{n=-\infty}^{+\infty} \delta(f - nf_s)$$

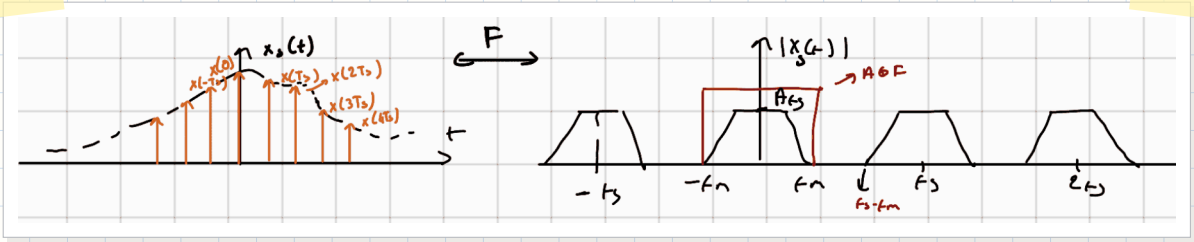


orijinal çizim, s. 1

$$x_s(t) = x(t) \cdot x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT_s)\,\delta(t - nT_s) \qquad (f(t)\,\delta(t - t_0) = f(t_0)\,\delta(t - t_0))$$

↑ Örneklenmiş işaret (Dirac katarı ile)



orijinal çizim, s. 1

$f_s - f_m > f_m$
$f_s > 2f_m$
$f_s \geq (f_N = 2f_m)$

$$x_s(t) = x(t) \cdot x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)}$$

$$\boxed{f(t)\,\delta(t - t_0) = f(t_0)\,\delta(t - t_0)}$$

$$\updownarrow F$$

$$X_s(f) = X(f) \otimes X_\delta(f)$$

$$= X(f) \otimes f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

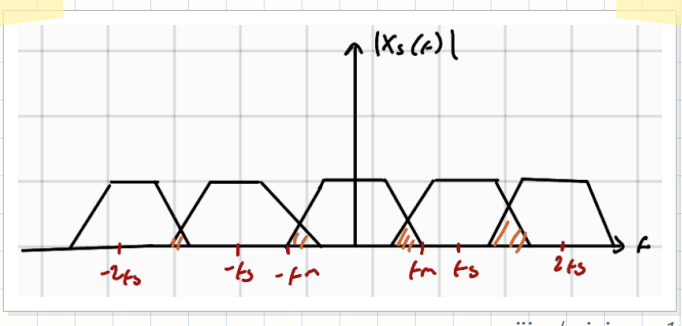
$$= f_s \sum_{n=-\infty}^{+\infty} X(f) \otimes \delta(f - nf_s)$$

$$\boxed{X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)}$$

$$\boxed{X(f) \otimes \delta(f - f_0) = X(f - f_0)}$$

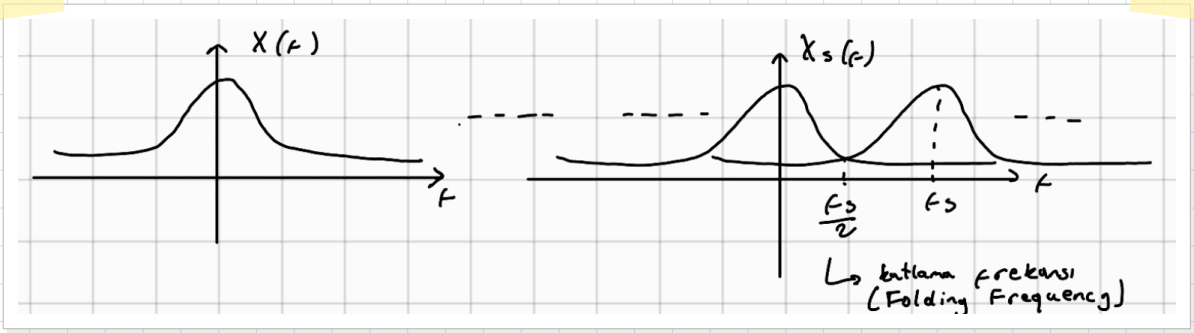
↑ sbt

$f_s < 2f_m$ olursa Örtüşme (Aliasing)



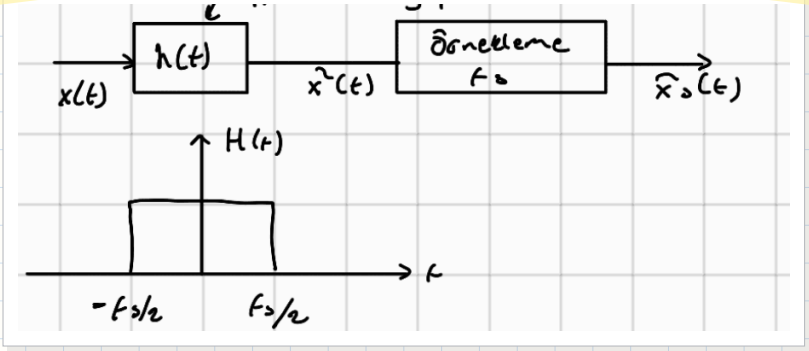
orijinal çizim, s. 1

orj. s. 2 Pratikte $x(t)$ zamanda sınırlı, frekansta sınırlı değil (not band limited)



orijinal çizim, s. 2

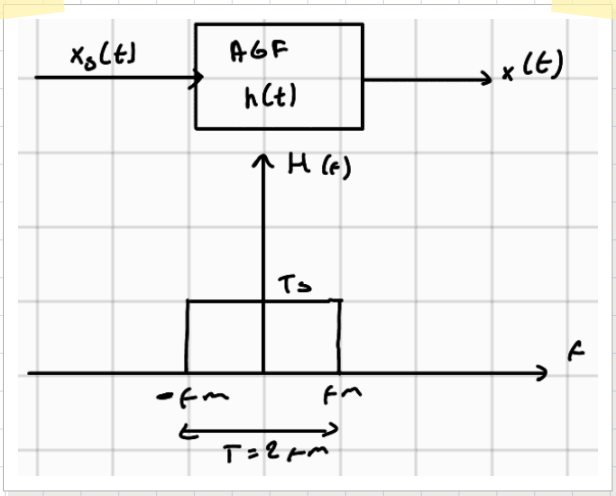
Çözüm: Örtüşme engelleyici filtre
Anti-aliasing filter



orijinal çizim, s. 2

İşareti yeniden oluşturma

(Signal reconstruction)



orijinal çizim, s. 2

$$H(f) = A\,\Pi\left(\frac{f}{T}\right) = T_s\,\Pi\left(\frac{f}{2f_m}\right)$$

$\downarrow F^{-1}$

$$h(t) = 2f_m T_s \,\mathrm{sinc}(2f_m t)$$ ← önemli

Eğer $f_s = 2f_m \rightarrow 1$ olur

$$A\,\Pi\left(\frac{t}{T}\right) \overset{F}{\longleftrightarrow} AT\,\mathrm{sinc}(fT)$$
$$AT\,\mathrm{sinc}(tT) \overset{F}{\longleftrightarrow} A\,\Pi\left(\frac{f}{T}\right)$$

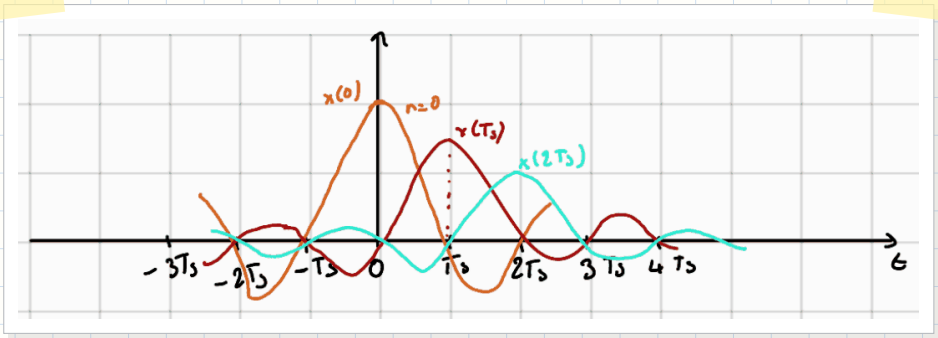
$$\hat{x}(t) = x_s(t) \otimes h(t)$$
$$= h(t) \otimes x_s(t) = h(t) \otimes \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$
$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\underbrace{[h(t) \otimes \delta(t - nT_s)]}_{h(t-nT_s)}$$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,h(t - nT_s)$$
$$= 2f_m T_s \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}[2f_m(t - nT_s)]$$

Üçgen varsa linear interpolasyon
Π varsa kare darbelerle temsil edilir

(yeniden oluşturma denklemi) Reconstruction eq
(İdeal interpolasyon //) ideal interpolation

orj. s. 3



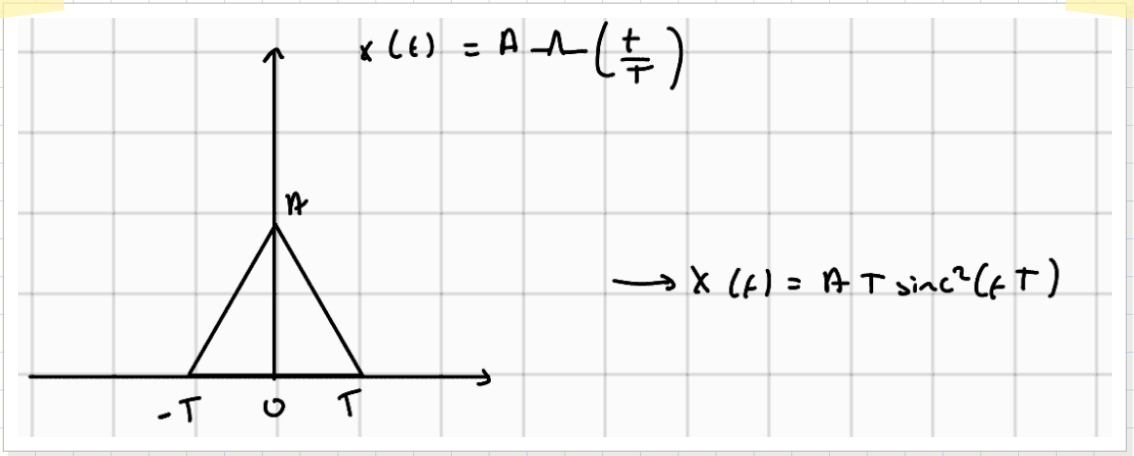
orijinal çizim, s. 3

Örnek: Lathi; Prob. 6.1-4

A signal $x(t) = \mathrm{sinc}^2(5t)$ is sampled (using uniformly spaced impulses) at a rate of:
i) 5 Hz, ii) 10 Hz, iii) 20 Hz

For each of three cases

a) Sketch the sampled signal
b) Sketch the spectrum of the sampled signal
c) Explain whether you can recover the signal $x(t)$ from the sampled signal
d) If the sampled signal is passed through an ideal LPF of BW 5 Hz, Sketch the spectrum of the output signal.

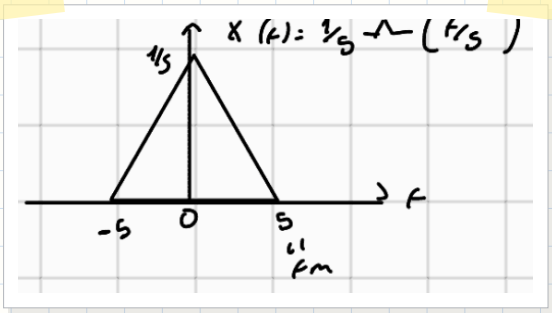


orijinal çizim, s. 3

$$AT\,\mathrm{sinc}^2(tT) \overset{F}{\longleftrightarrow} A\Lambda\left(\frac{f}{T}\right)$$

$$x(t) = \mathrm{sinc}^2(5t) \overset{F}{\longleftrightarrow}$$

$$A = \frac{1}{T} = \frac{1}{5}$$



orijinal çizim, s. 3

① $f_s = 5$ Hz, $T_s = \frac{1}{f_s} = \frac{1}{5}$ sn

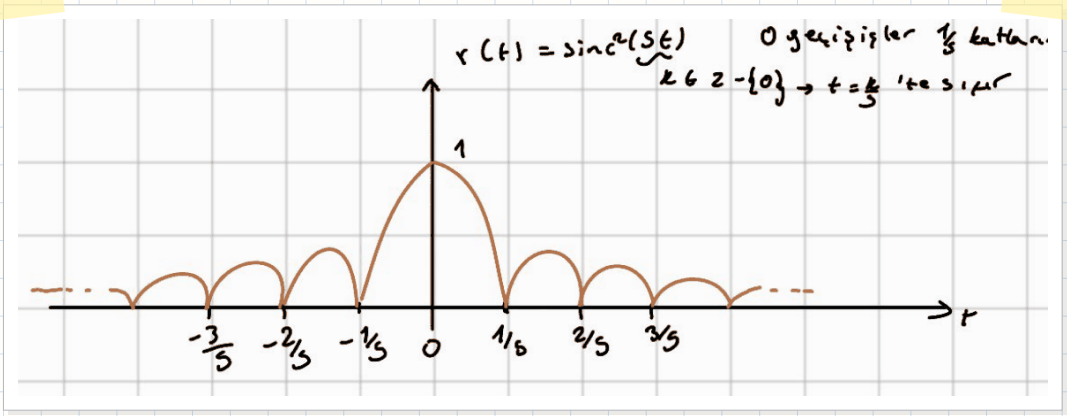
$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\underbrace{T_s}_{1/5})$$

$$x_s(t) = x(t)\,x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

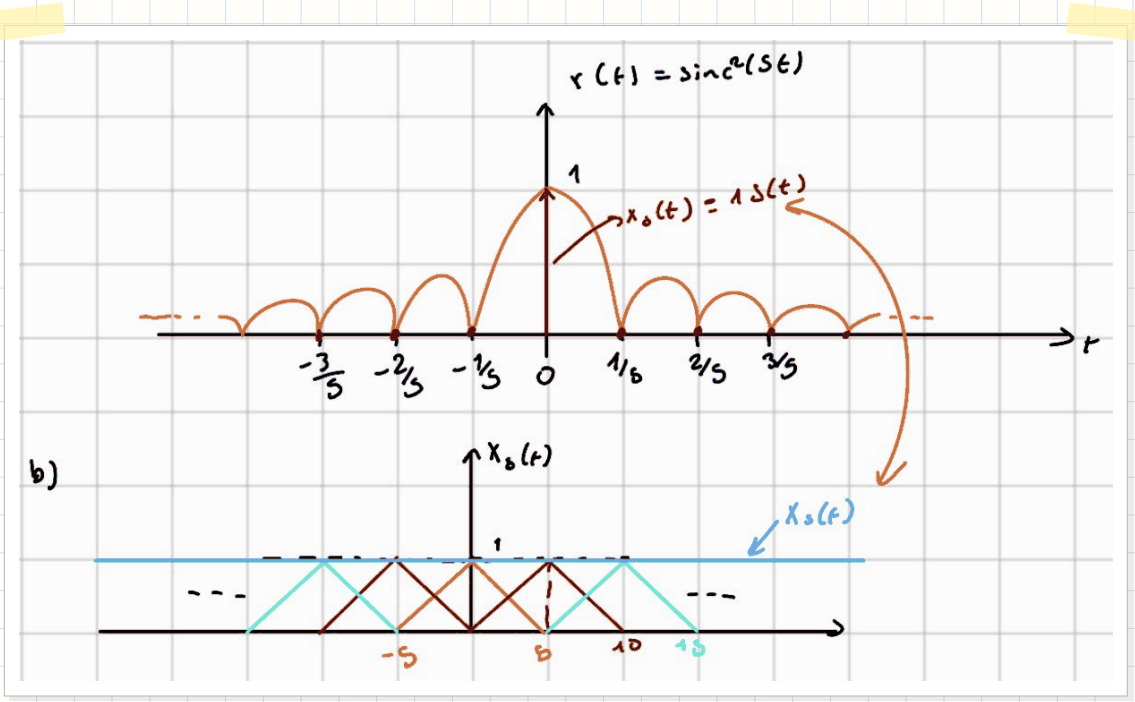
orj. s. 4

a)



orijinal çizim, s. 4

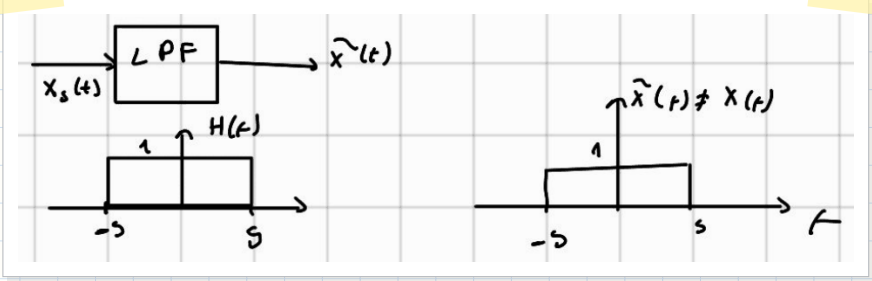
$$\text{sinc}\, k = \frac{\sin(\pi k)}{\pi k} = 0 \qquad k \in Z - \{0\}$$



orijinal çizim, s. 4

c) No, $f_s = 5$ Hz $< 2f_m = 10$ Hz

d)



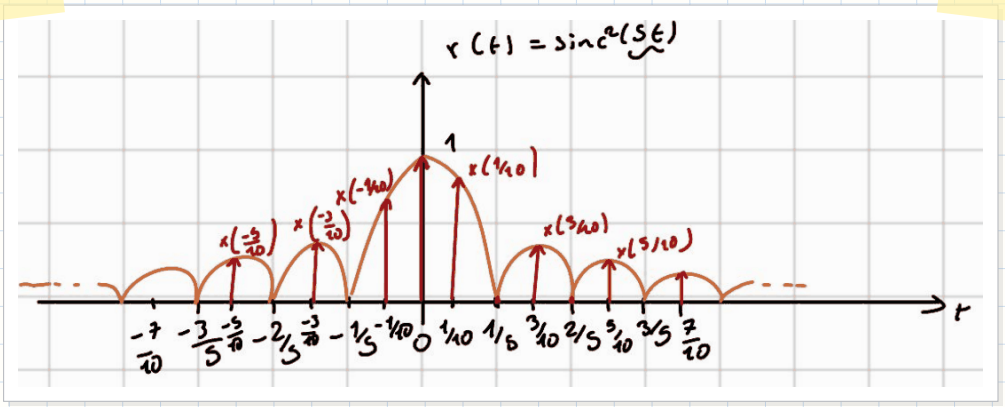
orijinal çizim, s. 4

ⓘⓘ $f_s = 10$ Hz, $T_s = \frac{1}{f_s} = \frac{1}{10}$ sn

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\underbrace{T_s}_{1/10})$$

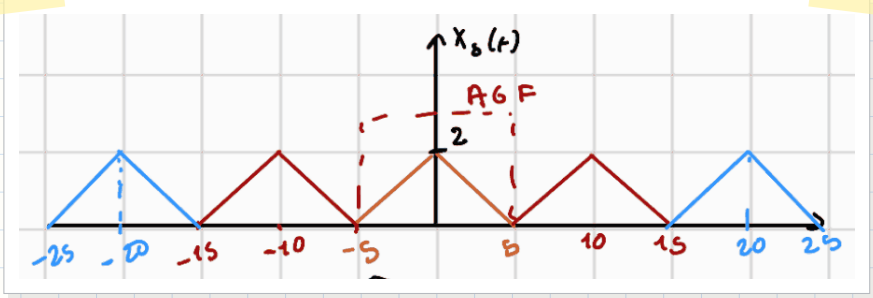
$$x_s(t) = x(t)x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(f) = \underset{10}{f_s} \sum_{n=-\infty}^{\infty} X(f - n\underset{10}{f_s})$$



orijinal çizim, s. 4

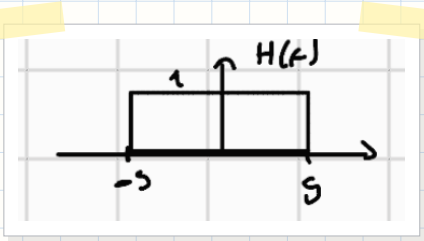
orj. s. 5



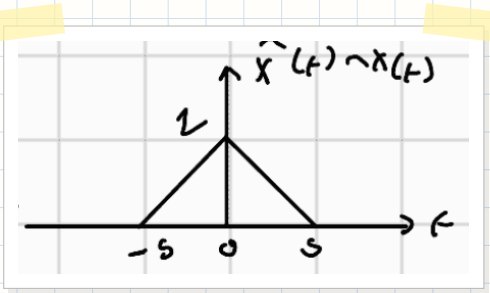
orijinal çizim, s. 5

c) Yes, $f_s = 10$ Hz $= 2f_m = 10$ Hz

d)



orijinal çizim, s. 5



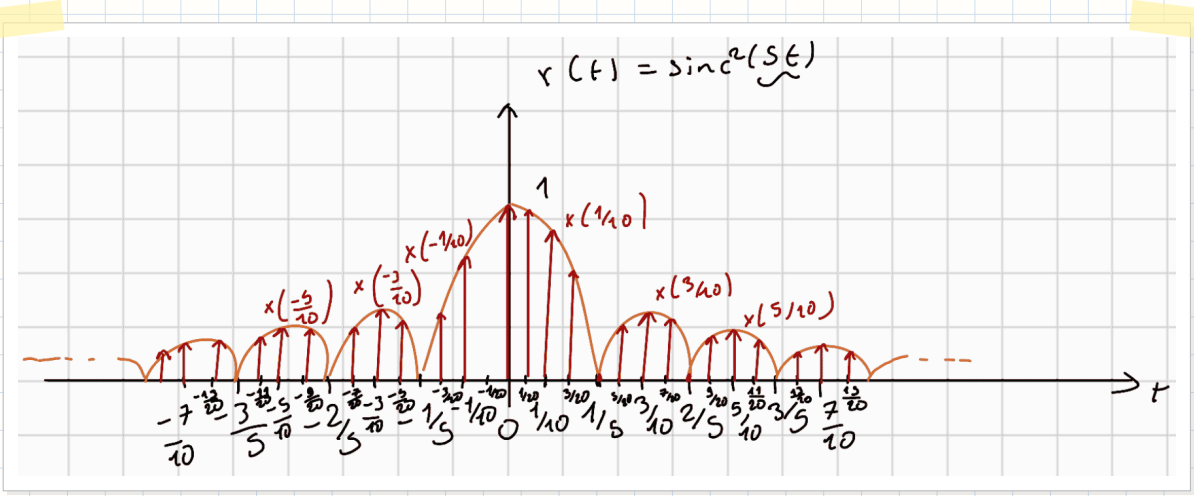
orijinal çizim, s. 5

(iii) $f_s = 20$ Hz, $T_s = \frac{1}{f_s} = \frac{1}{20}$ sn

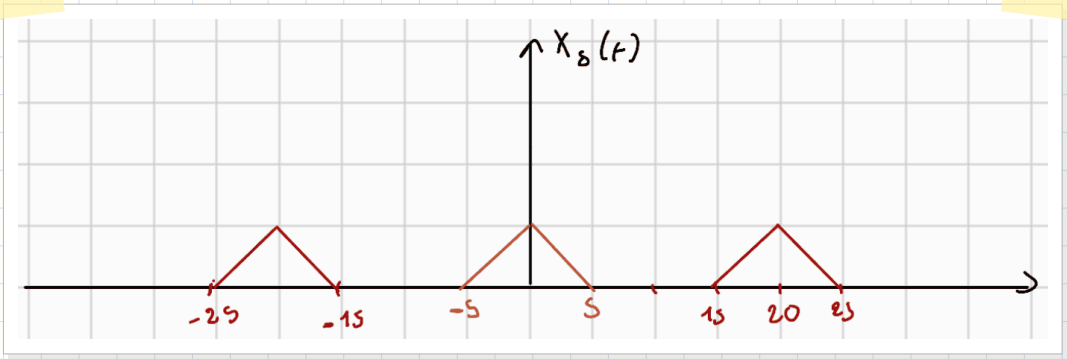
$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\underbrace{T_s}_{1/20})$$

$$x_s(t) = x(t)\,x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

$$X_s(f) = \underset{20}{f_s} \sum_{n=-\infty}^{\infty} X(f - n\underset{20}{f_s})$$



orijinal çizim, s. 5

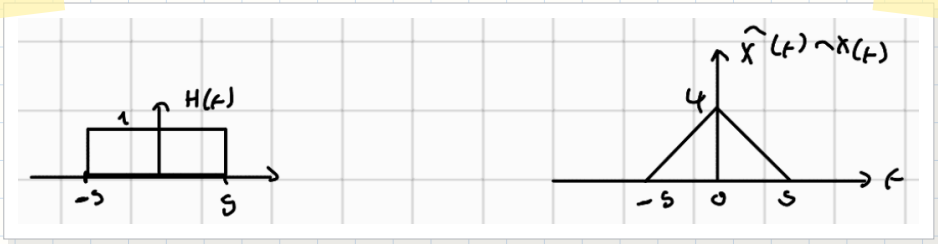


orijinal çizim, s. 5

orj. s. 6

c) Yes, $f_s = 20\ Hz > 2f_M = 10\ Hz$

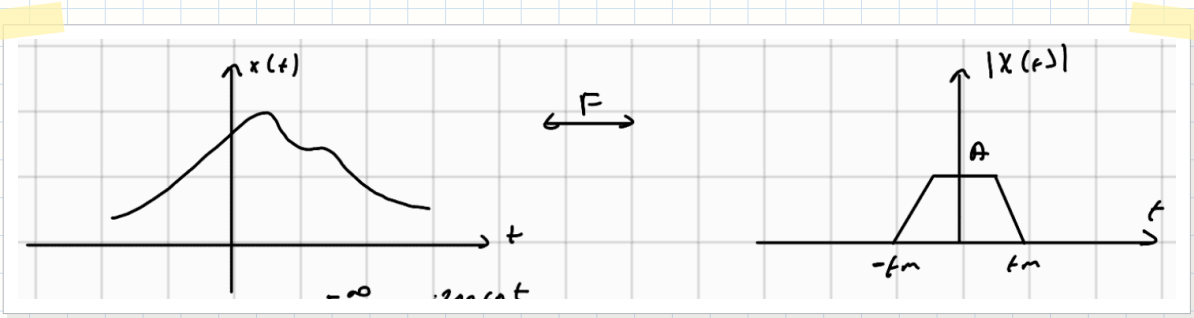
d)



orijinal çizim, s. 6

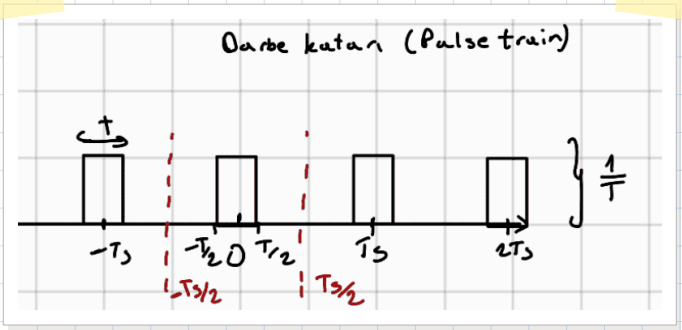
Pratik örnekleme

(Natural sampling)

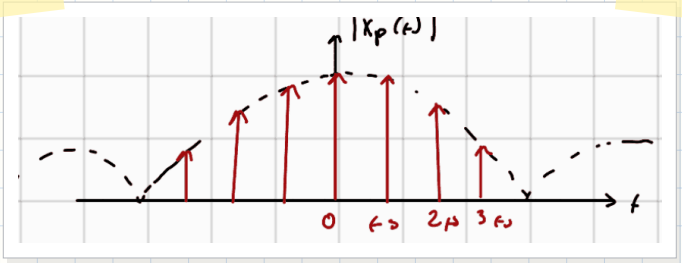


orijinal çizim, s. 6

$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

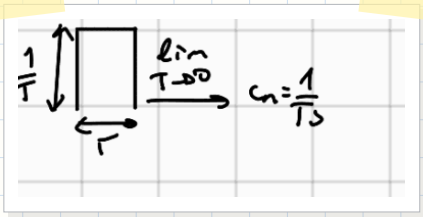


orijinal çizim, s. 6



orijinal çizim, s. 6

$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_s t}$$



orijinal çizim, s. 6

$$c_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} x_p(t)\, e^{-j2\pi n f_s t}\, dt = \frac{1}{T_s}\int_{-T/2}^{T/2} \frac{1}{T} e^{-j2\pi n f_s t}\, dt$$

$$= \frac{1}{T_s}\cdot\frac{1}{T}\cdot \left.\frac{e^{-j2\pi n f_s t}}{-j2\pi n f_s}\right|_{-T/2}^{T/2}$$

$$= \frac{1}{T_s}\cdot\frac{1}{T}\cdot \boxed{\frac{e^{-j\pi n f_s T} - e^{j\pi n f_s T}}{-j2\pi n f_s}}$$

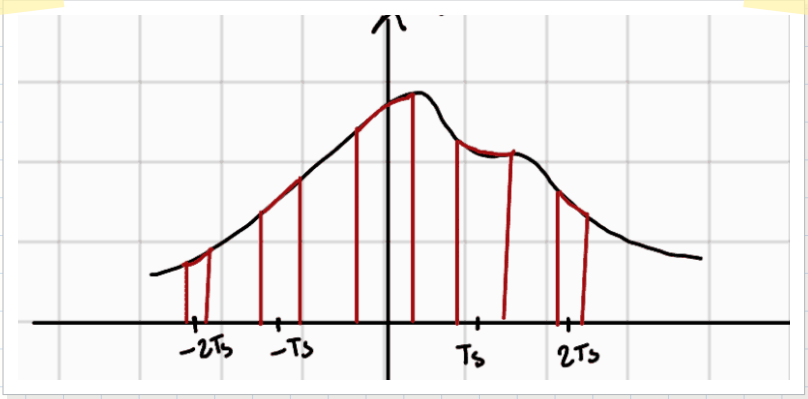
$$= \frac{1}{T_s}\frac{\sin(\pi n f_s T)}{\pi n f_s T} = \frac{1}{T_s}\operatorname{sinc}(n f_s T)$$

$$c_n = \frac{1}{T_s}\operatorname{sinc}(n f_s T)$$

$$X_p(f) = \sum_{n=-\infty}^{\infty} c_n\, \delta(f - n f_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \operatorname{sinc}(n f_s T)\, \delta(f - n f_s)$$

Darbe Genlik mod. (DGM) deniliyor ama bu değil!!! sonra

Pulse amplitude mod (PAM)



orijinal çizim, s. 6

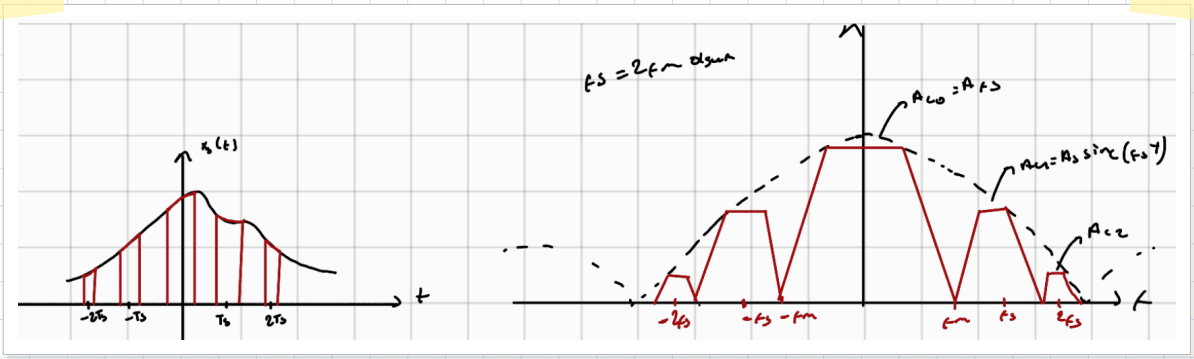
$$x_s(t) = x(t)\,x_p(t) = x(t)\sum_{n=-\infty}^{\infty} c_n\, e^{j2\pi n f_s t}$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} c_n\, x(t)\, e^{j2\pi n f_s t}$$

$$\downarrow F \qquad c_n = f_s \cdot \mathrm{sinc}(n f_s T)$$

$$X_s(f) = \sum_{n=-\infty}^{\infty} c_n\, X(f - n f_s)$$

orj. s. 7



orijinal çizim, s. 7

$$x_s(t) \longleftrightarrow X_s(f) = X(f) \otimes X_p(f) = X(f) \otimes \sum_{n=-\infty}^{\infty} c_n\, \delta(f - n f_s)$$

$$= \sum_{n=-\infty}^{+\infty} c_n\, X(f - n f_s)$$

Örnek: Lathi, Pr. 6.1-3

A signal $x(t)$ band limited to B Hz is sampled by a periodic pulse train $p_{T_s}(t)$ made up of a rectangular pulse of width $\frac{1}{8B}$ seconds (centered at the origin) repeating at the Nyquist rate ($2B$

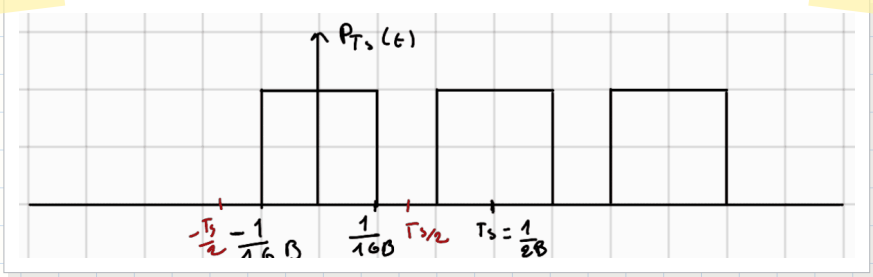
pulses per second). Show that the sampled signal $x_s(t)$ is given by

$$x_s(t) = \frac{1}{4}x(t) + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{4}\right) x(t)\cos(n\omega_s t), \qquad \omega_s = 2\pi f_s = 4\pi B$$

Show that the signal $x(t)$ can be recovered by passing $x_s(t)$ through an ideal LPF of BW B Hz and a gain of 4.

? okunamadı

$f_s = 2B$
$T_s = \frac{1}{2B}$



orijinal çizim, s. 7

$$x_s(t) = x(t) \cdot p_{T_s}(t)$$

$$p_{T_s}(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{4}\right) \cos(n\omega_s t) \qquad \omega_s = 2\pi f_s = 4\pi B$$

böyle olduğunu ispatlamamız gerek

$$p_{T_s}(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_s t) + b_n \sin(n\omega_s t)\,]$$

$$\begin{aligned} a_0 &= \frac{1}{T_s}\int_{T_s} x(t)\,dt = \frac{1}{T_0}\int_{-T_s/2}^{T_s/2} x(t)\,dt \\ &= 2B\int_{-1/16B}^{1/16B} 1\,dt = 2B \cdot \frac{2}{16B} \\ &= \frac{1}{4} \end{aligned}$$

orj. s. 8 $b_n = 0$ çünkü fonksiyon çift (p_{T_s} çift $\rightarrow b_n = 0$.)

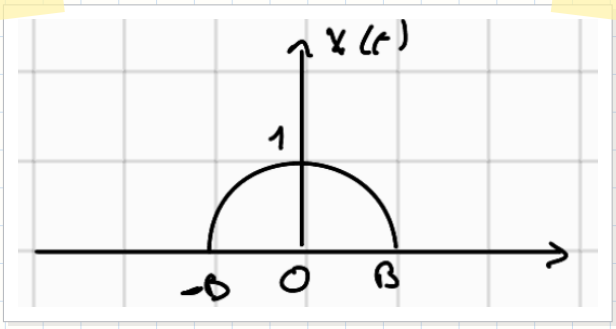
$$\begin{aligned} a_n &= \frac{2}{T_s}\int_{-T_s/2}^{T_s/2} p_{T_s}(t)\cos(n\omega_s t)\,dt = 4B\int_{-1/16B}^{1/16B} 1\cdot\cos(n\omega_s t)\,dt \\ &= 4B\,\frac{\sin(n\omega_s t)}{n\,\omega_s}\Bigg|_{-1/16B}^{1/16B} \\ &= \frac{1}{n\pi}\left[\sin\left(n\cdot 4\pi B\cdot\frac{1}{16B}\right) - \sin\left(n\cdot 4\pi B\cdot\frac{-1}{16B}\right)\right] \end{aligned}$$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{4}\right)$$

$$a_0 = \frac{\pi}{4}^{1} \rightarrow p_{T_s}(t) = \underbrace{\frac{1}{4}}_{a_0} + \underbrace{\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{4}\right)}_{a_n}\cos(n\omega_s t)$$

$$x_s(t) = x(t)\,p_{T_s}(t) = \frac{1}{4}x(t) + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{4}\right)\underbrace{\cos(n\omega_s t)}_{e^{j2\pi n f_s t}}x(t)$$

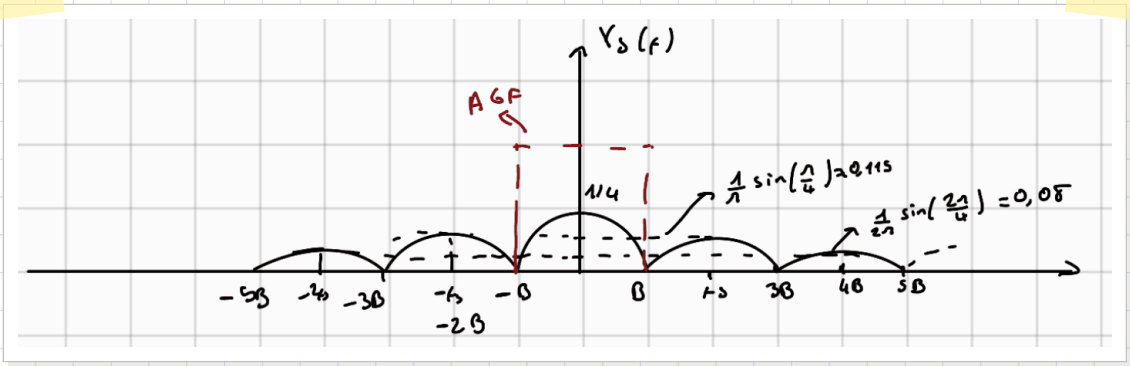
b)



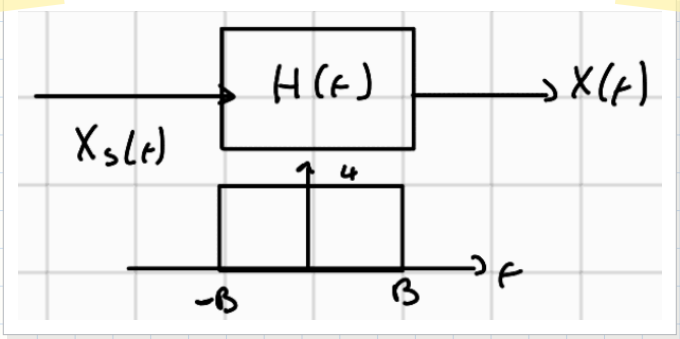
orijinal çizim, s. 8

olsun

$$X_s(f) = \frac{1}{4}X(f) + \frac{1}{2}\frac{2}{n\pi}\sin\left(\frac{n\pi}{4}\right)\left[X(f-nf_s) + X(f+nf_s)\right]$$



orijinal çizim, s. 8



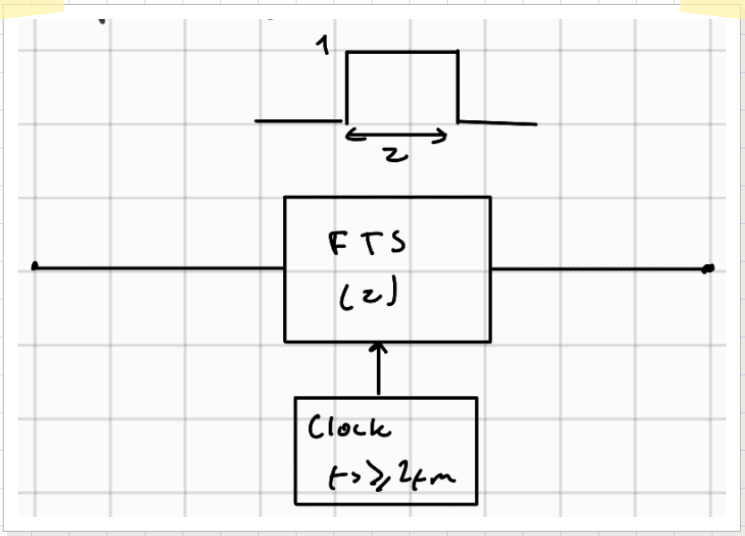
orijinal çizim, s. 8

[1] Claude notu: muhtemelen $a_0 = \frac{1}{4}$ olmalı.

orj. s. 9

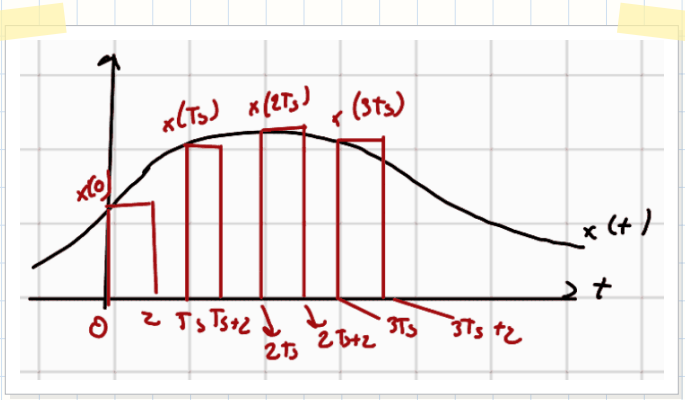
Düz Tepeli örnekleme

(Flat top sampling) (FTS)



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tepeyi takip etmiyor



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Sample and hold circuit
(Örnekle ve tut devresi)

$x_s(t)$: Darbe Genlik mod (DGM)
Pulse amplitude mod