Sayısal Haberleşme

İkili işaretleşmede hata olasılığı ve MAP kriteri

Orijinal not: Ders-9.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1

$$r(t) = s_i(t) + n(t) \longrightarrow \boxed{\text{MF}}$$

$$P_b = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}(1-\rho)}\right)$$

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2 \, dt$$

$$\rho = \frac{1}{E_b}\int_0^T s_1(t)\, s_2(t)\, dt$$

→ E_b: ortalama bit enerjisi / average energy per bit
→ ρ: karşı ilişki katsayısı, $-1 \le \rho \le 1$

$$\gamma_b = \frac{E_b}{N_0} \quad : \quad \text{Bit başına İGO (SNR per bit)}$$

$s_1(t) \perp s_2(t)$ ($s_1(t)$ ve $s_2(t)$ ortogonal) $\Rightarrow \rho = 0 \Rightarrow P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

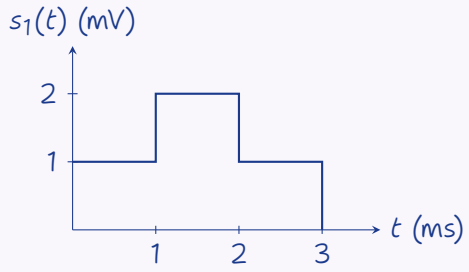
$s_1(t) = -s_2(t)$ (" " " zıt işaretler (antipodal)) $\Rightarrow \rho = -1 \Rightarrow P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$$E_b = \sum_{i=1}^{2} E_i P(s_i) = E_1 P(s_1) + E_2 P(s_2)$$

E_i: $s_i(t)$'nin enerjisi

Örnek

Consider a binary communication system that receives equally likely signals $s_1(t)$ and $s_2(t)$ plus AWGN. Assume that the receiving filter is a MF, and the noise PSD N_0 is 10^{-12} Watt/Hz. Use the values of the received signal voltage and time shown in the figure below to compute bit error probability.



$s_2(t) = -s_1(t)$

orj. s. 2

$$P(s_1) = P(s_2) = \frac{1}{2}$$
$$N_0 = 10^{-12}\ \text{W}$$
$$s_2 = -s_1 \rightarrow E_b = E_{s_1} = E_{s_2}$$

$$E_b = E_{s_1} = \int_0^T s_1^2(t)\,dt = \int_0^{10^{-6}} (10^{-3})^2\,dt + \int_{10^{-6}}^{2\cdot10^{-6}} (2\cdot10^{-3})^2\,dt + \int_{2\cdot10^{-6}}^{3\cdot10^{-6}} (10^{-3})^2\,dt = 6\cdot10^{-12}\ \text{J}$$

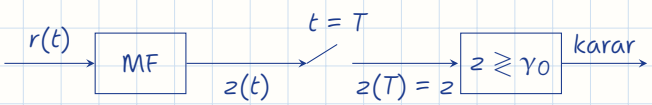
$$P_b = Q\left(\sqrt{\frac{2\cdot6\cdot10^{-12}}{10^{-12}}}\right) = Q(\sqrt{12}) = Q(3{,}46) = 3\cdot10^{-4} \quad \text{(Tablodan)}$$
$$= \frac{3}{10\,000}$$

İkili işaretleşmenin Hata olasılığı Performansı

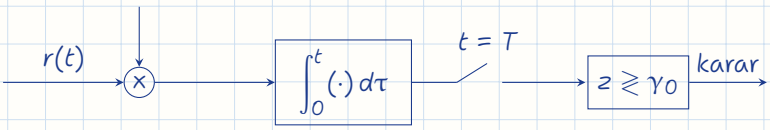
Tek kutuplu işaretleşme (Unipolar signaling)

$$\left.\begin{array}{l} \text{"1"} \rightarrow s_1(t) = A, \quad 0 \le t \le T,\ A > 0 \\ \text{"0"} \rightarrow s_2(t) = 0, \quad 0 \le t \le T \end{array}\right\} \Rightarrow \rho = \frac{1}{E_b}\int_0^T s_1(t)\,s_2(t)\,dt = 0$$





$$h(t) = s_1(T-t) - s_2(T-t) = s_1(T-t), \quad 0 \le t \le T$$
$$s_1(t) - s_2(t) = s_1(t) = A, \quad 0 \le t \le T$$



$$r(t) = s_i(t) + n(t)$$

$$z(t) = r(t) \circledast h(t) \xrightarrow{t=T} z = z(T) = \underbrace{r(t) \circledast h(t)}_{a_i(t)+n_0(t)}\Big|_{t=T}$$

$$\Rightarrow z = z(T) = a_i(T) + n_0(T) = a_i + n_0$$

$$a_i = a_i(T) = s_i(t) \circledast h(t)\Big|_{t=T}, \qquad \boxed{i = 1,2}$$

$$a_i = a_i(T) = \int_0^T s_i(t)\,[s_1(t) - s_2(t)]\,dt \qquad i = 1,2$$

← önemli

orj. s. 3

Tek kutup işaretleşme $\Rightarrow a_i = \int_0^T s_i(t)\,[s_1(t) - \underbrace{s_2(t)}_{0}]\,dt \qquad i = 1,2$

$$\left.\begin{aligned} a_1 &= \int_0^T s_1^2(t)\,dt = \int_0^T A^2\,dt \\ a_2 &= \int_0^T s_2(t)\,s_1(t)\,dt = 0 \end{aligned}\right\} \quad \gamma_0 = \frac{a_1 + a_2}{2} = \frac{A^2 T}{2}$$

$$\text{Eğer } z > \gamma_0 = \frac{A^2T}{2} \Rightarrow \text{karar} = s_1(t)$$
$$z < \gamma_0 \Rightarrow \text{karar} = s_2(t)$$

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \int_0^T s_1^2(t)\,dt = A^2 T$$

$$P_b = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)$$

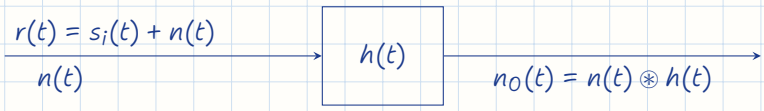
$$\Rightarrow E_b = E_1 \underset{1/2}{P(s_1)} + E_2 \underset{1/2}{P(s_2)} = \frac{A^2T}{2}$$

$$E_1 = \int_0^T s_1^2(t)\,dt = A^2T$$

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}(1-\rho)}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_b = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$



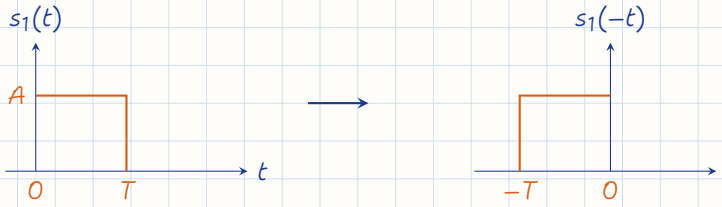
$$S_n(f) = \frac{N_0}{2} \frac{\text{Watt}}{\text{Hz}}$$

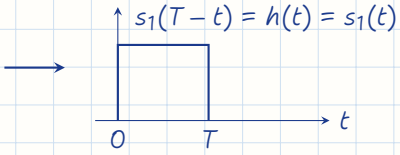
$$S_{n_0}(f) = S_n(f)\,|H(f)|^2$$

$$\sigma_0^2 = P_{n_0} = \int_{-\infty}^{\infty} S_{n_0}(f)\,df = \frac{N_0}{2} \underbrace{\int_{-\infty}^{\infty} |H(f)|^2\,df}_{E_h} = \frac{N_0}{2} E_h$$

$$\boxed{\sigma_0^2 = \frac{N_0}{2} E_h}$$

Tek kutuplu da $h(t) = s_1(T - t)$





$$h(t) = s_1(t) \Rightarrow E_h = E_{s_1} = E_1 = A^2 T$$

$$\sigma_0^2 = \frac{N_0}{2} E_h = \frac{N_0}{2} A^2 T$$

$$P_b = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q\left(\frac{A^2 T - 0}{2\sqrt{\frac{N_0}{2} A^2 T}}\right)$$

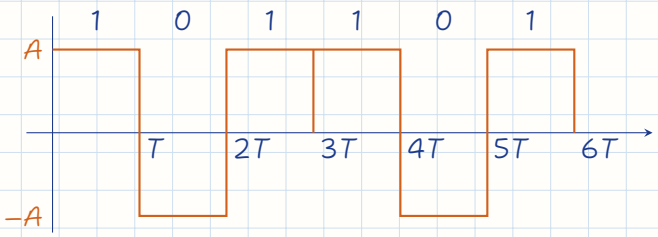
$$= Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

orj. s. 4

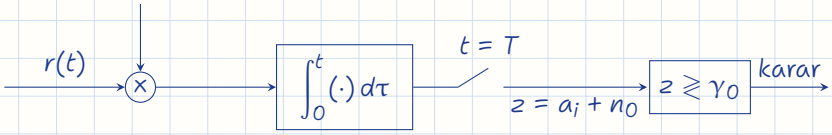
Çift kutuplu işaretleşme (Bipolar signaling)

$$\text{"1"} \rightarrow s_1(t) = A, \quad 0 \le t \le T$$

$$\text{"0"} \rightarrow s_2(t) = -A, \quad 0 \le t \le T$$



$s_1(t) - s_2(t) = 2A, \quad 0 \le t \le T$



$$a_i = \int_0^T s_i(t)\underbrace{[s_1(t) - s_2(t)]}_{2A}\,dt$$

$$\left.\begin{aligned} a_1 &= \int_0^T \underbrace{s_1(t)}_{A}\, 2A\,dt = 2A^2T \\ a_2 &= \int_0^T \underbrace{s_2(t)}_{-A}\, 2A\,dt = -2A^2T \end{aligned}\right\} \quad \gamma_0 = \frac{a_1 + a_2}{2} = 0$$

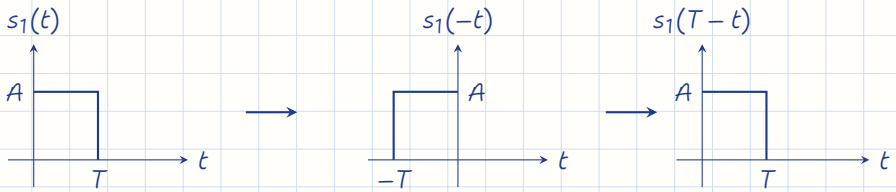
Eğer $z > 0 \Rightarrow$ karar $= s_1(t)$

$z < 0 \Rightarrow$ " $= s_2(t)$

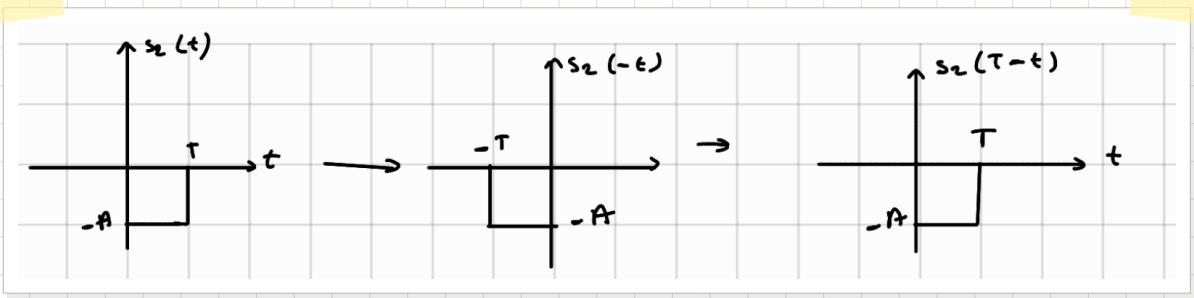
$$E_1 = E_2 = \int_0^T s_1^2(t)\,dt = A^2T$$

$$\Rightarrow E_b = \underset{A^2T}{E_1}\ \underset{1/2}{P(s_1)} + \underset{A^2T}{E_2}\ \underset{\cdot 1/2}{P(s_2)} = A^2T$$

$E_h = ??$

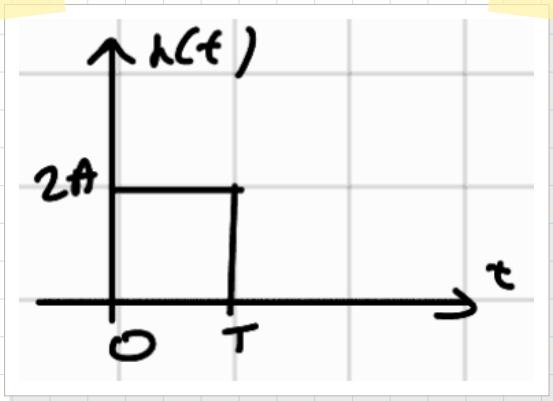


orj. s. 5



orijinal çizim, s. 5

$$h(t) = 2A, \quad 0 \le t \le T$$



orijinal çizim, s. 5

$$E_h = \int_0^T h^2(t)\,dt = \int_0^T 4A^2\,dt = 4A^2T$$

$$\sigma_0^2 = \frac{N_0}{2}E_h = \frac{N_0}{2}\,4A^2\,T = 2N_0A^2T$$

$$E_d = \int_0^T \underbrace{[s_1(t) - s_2(t)]}_{2A}^2\,dt = \int_0^T 4A^2\,dt = 4A^2T$$

$$\boxed{P_b = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)} = Q\left(\frac{2A^2T - (-2A^2T)}{2\sqrt{2N_0A^2T}}\right) = Q\left(\frac{4A^2T}{\sqrt{8N_0A^2T}}\right)$$

$$= Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\boxed{P_b = Q\left(\sqrt{\frac{E_b}{N_0}(1 - \underset{-1}{\rho})}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)} \quad \leftarrow \textit{önemli}$$

$$P_b = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{4A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Fig 3.14

> *Şekil:* Basılı şekil: tek kutuplu ve çift kutuplu işaretleşmenin bit hata başarımı ($P_b - E_b/N_0$ eğrileri); dijitalleştirilmedi. Şekil üzerindeki el yazısı notlar: "Tek kutuplu işaretleşme", "Çift kutuplu", "3 dB".
> *(orijinal notta, s. 5)*

$$10\log_{10}\frac{2E_b}{N_0} = \underbrace{10\log_{10}2}_{3\ \text{dB}} + 10\log_{10}\frac{E_b}{N_0}$$

Çift kutuplu daha iyi performans sağlar.

orj. s. 6

Var yok haberleşmesi, ışık fiber haberleşmede kullanılır (tek kutuplu)

Örnek: Sklar, S.163, Prob 3.5

> **Şekil:** Basılı soru (Sklar Prob. 3.5): ±1 V'luk çift kutuplu ikili işaret, iki taraflı güç spektral yoğunluğu 10^{-3} W/Hz olan toplamsal beyaz Gauss gürültüsü, eşleşmiş süzgeçle algılama; $P_B \leq 10^{-3}$ için gönderilebilecek maksimum bit hızı isteniyor. *(orijinal notta, s. 6)*

Max bit rate?

$$\mathcal{R} = \frac{1 \text{ bit}}{T \text{ sn}}$$

$$\rightarrow P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \leq \underbrace{10^{-3}}_{0{,}001} \qquad E_b = A^2T = 1^2T = T$$

$$\sqrt{\frac{2E_b}{N_0}} \geq 3{,}08 \quad \text{(Tablodan)}$$

$$\Rightarrow \sqrt{\frac{2T}{N_0}} \geq 3{,}08 \Rightarrow \frac{2T}{N_0} \geq 9{,}4864$$

$$\Rightarrow T \geq \frac{N_0}{2} \cdot 9{,}4864 = 9{,}48 \cdot 10^{-3} \text{ sn}$$

$$\mathcal{R} = \frac{1}{T} \leq 105{,}41 \text{ bit/sn}$$

Örnek Sklar, Prob 3.14

> **Şekil:** Basılı soru (Sklar Prob. 3.14): NRZ ikili darbeler, güç zayıflaması 3 dB olan kablo, koherent algılama, veri hızı 56 kbit/s, $N_0 = 10^{-6}$ W/Hz; $P_B = 10^{-3}$ için vericide gereken minimum güç isteniyor. *(orijinal notta, s. 6)*

NRZ-L, attenuation 3 dB
Coherent detection
$\mathcal{R}$ = 56 kbit/sec, $N_0 = 10^{-6}$ Watt/Hz
min amount of power needed at Tx for $P_b = 10^{-3}$
Bipolar signaling $\Rightarrow P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 10^{-3}$

$$\left.\begin{aligned} E_b &= A^2T \\ \mathcal{R} &= \frac{1}{T} = 56000 \ \tfrac{\text{bit}}{\text{sn}} \end{aligned}\right\} \Rightarrow Q\left(\underbrace{\sqrt{\frac{2A^2(1/56\,000)}{10^{-6}}}}_{=3{,}1 \text{ tablodan}}\right) = 10^{-3}$$

$$\sqrt{\frac{2A^2(1/56\,000)}{10^{-6}}} = 3{,}1 \Rightarrow A^2 = 0{,}265 \text{ Watt} \approx 270 \text{ mW}$$

orj. s. 7



orijinal çizim, s. 7

$$10 \log_{10} P_T = 10 \log_{10} \mathcal{L} + 10 \log_{10} P_R$$

$$\mathcal{L}_d = 10 \log_{10} \frac{P_T}{P_R}$$

$$P_R = A_r^2 = 270 \text{ mW}$$

$$\begin{aligned} P_T &= P_R \cdot 10^{\mathcal{L}_{dB}/10} \\ &= 270 \text{ mW} \cdot 10^{3/10} \\ &= 540 \text{ mW} \end{aligned}$$

Likelihood Ratio Test and Maximum a Posteriori probability (MAP) Criterion

Muhtemellik oranı testi ve maksimum sonsal olasılık kriteri

$$P(s_1/z) \underset{H_2}{\overset{H_1}{\gtrless}} P(s_2/z) \qquad (H_i: \text{ Hipotez})$$

$$P(s_i/z) = \frac{f(z/s_i)\,P(s_i)}{f(z)} \qquad i = 1,2$$

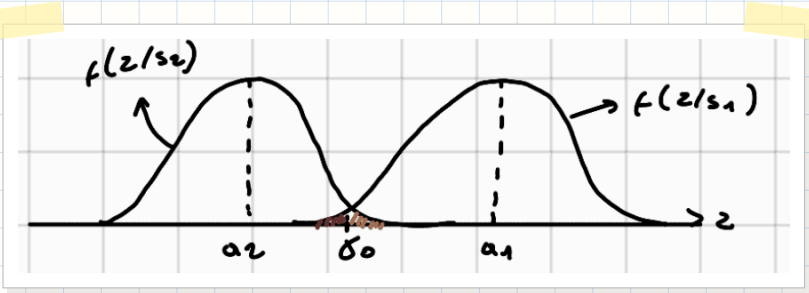
$$f(z) = \sum_{i=1}^{2} f(z/s_i)\,P(s_i) = f(z/s_1)\,P(s_1) + f(z/s_2)\,P(s_2)$$

$$f(z/s_1)\,P(s_1) \underset{H_2}{\overset{H_1}{\gtrless}} f(z/s_2)\,P(s_2)$$

$$\Lambda(z) = \frac{f(z/s_1)}{f(z/s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \qquad \text{likelihood ratio test}$$

$$\text{Eğer } P(s_1) = P(s_2) = \frac{1}{2} \Rightarrow \Lambda(z) = \frac{f(z/s_1)}{f(z/s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} 1$$

Maximum likelihood criterion
(En muhtemeli bulma kriteri)



orijinal çizim, s. 7

$$\Lambda(z) = \frac{\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(z-a_1)^2}{2\sigma_0^2}}}{\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(z-a_2)^2}{2\sigma_0^2}}} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

orj. s. 8

$$\Lambda(z) = \exp\underbrace{\left\{ -\frac{z^2 + a_1^2 - 2za_1}{2\sigma_0^2} + \frac{z^2 + a_2^2 - 2za_2}{2\sigma_0^2} \right\}}_{\frac{a_2^2 - a_1^2}{2\sigma_0^2} + \frac{2z(a_1 - a_2)}{2\sigma_0^2}} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

$$L(z) = \ln \Lambda(z) = \frac{a_2^2 - a_1^2}{2\sigma_0^2} + \frac{z(a_1 - a_2)}{\sigma_0^2} \underset{H_2}{\overset{H_1}{\gtrless}} \ln \frac{P(s_2)}{P(s_1)}$$

$$\frac{z(a_1 - a_2)}{\sigma_0^2} \underset{H_2}{\overset{H_1}{\gtrless}} \ln \frac{P(s_2)}{P(s_1)} + \frac{a_1^2 - a_2^2}{2\sigma_0^2}$$

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{\sigma_0^2}{a_1 - a_2} \ln \frac{P(s_2)}{P(s_1)} + \frac{\sigma_0^2}{a_1 - a_2} \frac{a_1^2 - a_2^2}{2\sigma_0^2}$$

$a_1 + a_2$ kalır

$$\boxed{z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{\sigma_0^2}{a_1 - a_2} \ln \frac{P(s_2)}{P(s_1)} + \frac{a_1 + a_2}{2}} = \gamma_0$$

Eğer $P(s_1) = P(s_2) = \frac{1}{2} \implies \gamma_0 = \frac{a_1 + a_2}{2}$

$$P(h/s_1) = \int_{-\infty}^{\gamma_0} f(z/s_1)\, dz = \frac{1}{\sqrt{2\pi}\sigma_0} \int_{-\infty}^{\gamma_0} e^{-\frac{(z-a_1)^2}{2\sigma_0^2}}\, dz$$

$$= 1 - \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\gamma_0}^{\infty} e^{-\frac{(z-a_1)^2}{2\sigma_0^2}}\, dz$$

$$= 1 - \underbrace{\frac{1}{\sqrt{2\pi}\sigma_0} \int_{\frac{\gamma_0 - a_1}{\sigma_0}}^{\infty} e^{-t^2/2}\, \sigma_0\, dt}_{Q\left(\frac{\gamma_0 - a_1}{\sigma_0}\right)}$$

$$\frac{z - a_1}{\sigma_0} = t \qquad z = \gamma_0 \Rightarrow t = \frac{\gamma_0 - a_1}{\sigma_0}$$

$$z = \infty \rightarrow t = \infty$$

$$\Rightarrow z = \sigma_0 t + a_1$$
$$dz = \sigma_0\, dt$$

$$\Rightarrow \boxed{P(h/s_1) = 1 - Q\left(\frac{\gamma_0 - a_1}{\sigma_0}\right)}$$

$$P(h/s_2) = \int_{\gamma_0}^{\infty} f(z/s_2)\, dz$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_0} \int_{\gamma_0}^{\infty} e^{-\frac{(z-a_2)^2}{2\sigma_0^2}}\, dz$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_0} \int_{\frac{\gamma_0 - a_2}{\sigma_0}}^{\infty} e^{-t^2/2}\, \sigma_0\, dt}_{Q\left(\frac{\gamma_0 - a_2}{\sigma_0}\right)}$$

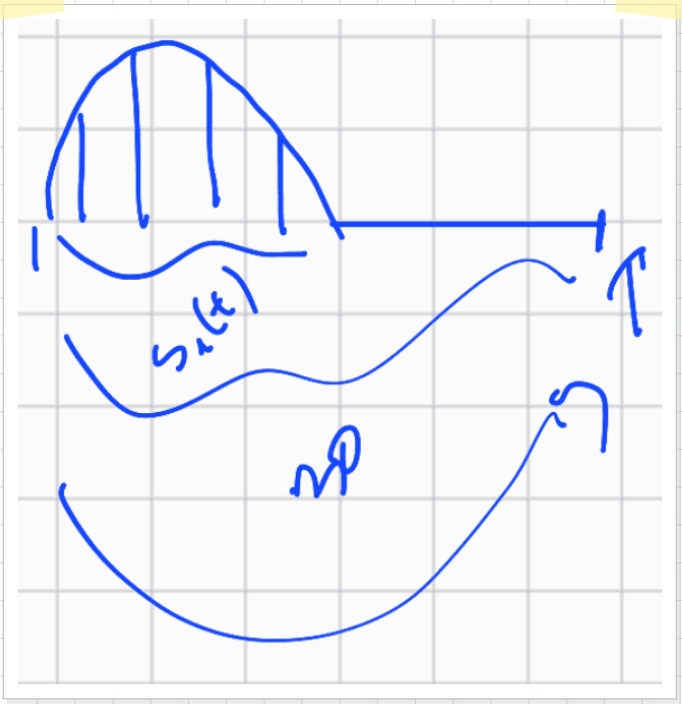
orj. s. 9

$$P(h/s_2) = Q\left(\frac{\gamma_0 - a_2}{\sigma_0}\right)$$

$$P_b = P(h/s_1) \cdot P(s_1) + P(h/s_2) \cdot P(s_2)$$

$$\boxed{P_b = \underbrace{\left[1 - Q\left(\frac{\gamma_0 - a_1}{\sigma_0}\right)\right]}_{Q\left(\frac{a_1 - \gamma_0}{\sigma_0}\right)} P(s_1) + Q\left(\frac{\gamma_0 - a_2}{\sigma_0}\right) P(s_2)} \quad \leftarrow \text{önemli}$$

$$Q(-x) = 1 - Q(x)$$



orijinal çizim, s. 9



orijinal çizim, s. 9