Sayısal Haberleşme

Olasılık teorisi, rastgele değişkenler ve prosesler

Orijinal not: Ders2.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1

Olasılık teorisi, Rastgele Değişkenler ve Rastgele Prosesler

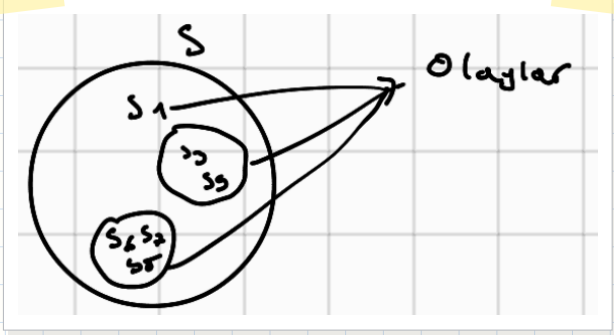
Deney (experiement): Zar atışı

Sonuç (outcome)

Örnek uzayı (sample space) : $S = \{1,2,3,4,5,6\}$

Olay (event) : $A = \{\text{Çift sayılar}\} = \{2,4,6\}$

$A \subset S$



orijinal çizim, s. 1

$$P(\overset{\text{olay}}{A}) = \sum_{\forall s_i \in A} P(s_i) = P(2) + P(4) + P(6)$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

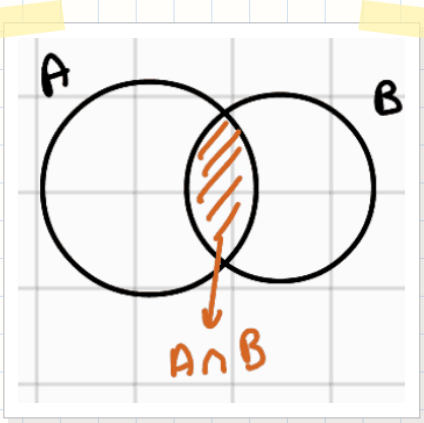
Eşit olasılıklı sonuçlar için

$$P(A) = \frac{|A|}{|S|} = \frac{A\text{'nın eleman sayısı}}{S\text{'nin } \text{''} \text{''}} = \frac{3}{6} = \frac{1}{2}$$

Bir olayın olması için olaydaki tek bir sonucun olması yeterli.

Kesin olay (Certain event) = $S \rightarrow P(S) = 1$

İmkansız olay (Impossible event) = $\emptyset \rightarrow P(\emptyset) = 0$



orijinal çizim, s. 1

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$A \cap B = \emptyset \Rightarrow$ A ve B kesişimsiz olaylar
(mutually exclusive, disjoint)

$$A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{0}$$

$$P(A \cup B) = P(A) + P(B)$$

$A' = \{a \mid a \in S,\ a \notin A\}$ uzayın A dışındaki tüm elemanlar

$$A \cup A' = S$$

$$A \cap A' = \emptyset$$

orj. s. 2

$$S = A \cup A'$$

$$P(S) = P(A \cup A') = P(A) + P(A') - \underbrace{P(A \cap A')}_{\emptyset}$$

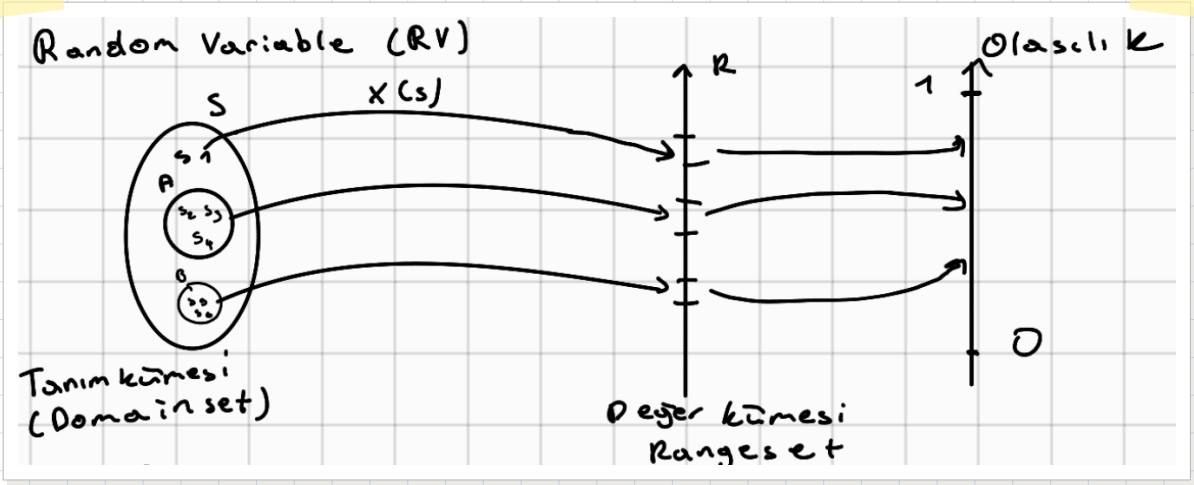
$$1 = P(A) + P(A')$$

$$\Rightarrow P(A') = 1 - P(A)$$

$A \perp\!\!\!\perp B \iff P(A \cap B) = P(A) \cdot P(B)$ (Bağımsızsa (independent))

Rastgele Değişken (RD)

$X(s)$: RD



orijinal çizim, s. 2

Rastgele değişken

$$X(s) = x$$

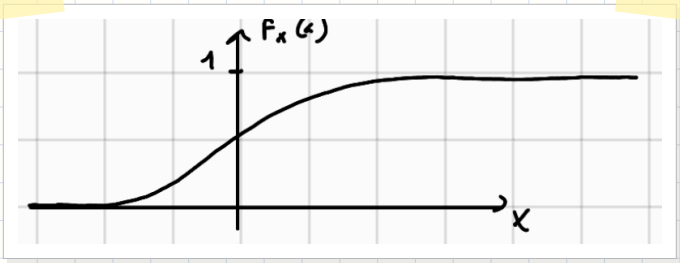
x : X RD'nin belli bir değeri.

örnek Y, y

Olasılık Dağılım Fonksiyonu (ODF)

Cumulative Distribution function - CDF

$$F_X(x) = \overset{\text{olasılık}}{P}(X \le x)$$



orijinal çizim, s. 2

- $F_X(-\infty) = 0$
- $F_X(+\infty) = 1$
- $0 \le F_X(x) \le 1$
- $x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$ Azalmayan fonksiyon (Nondecreasing function)

Olasılık Yoğunluk Fonk. (OYF)

Probability Density Function (PDF)

Bir rastgele değişkenin belli bir bölgede olmasının olasılığını, altında kalan alanla gösteren fonksiyona denir.

$f_X(x)$ OYF

orj. s. 3

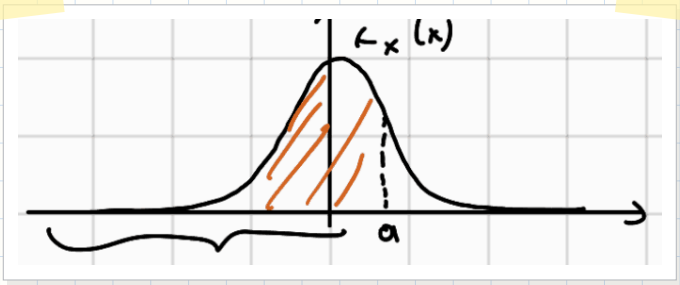
$$P(X \in A) = \int_A f_X(x)\,dx$$

$$P(X \le x) = \int_{-\infty}^{x} \overset{\text{OYF}}{f_X(t)}\,dt = \overset{\text{ODF}}{F_X(x)}$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
- $\forall x \quad f_X(x) \ge 0$ — azalmayan fonksiyonun türevi pozitiftir

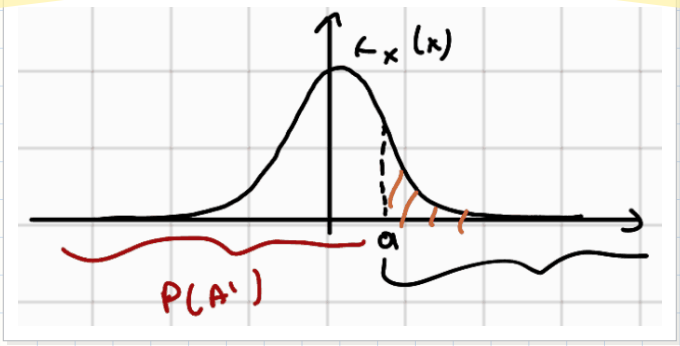
1) $P(X \le a) = \int_{-\infty}^{a} f_X(u)\,du = F_X(a)$



orijinal çizim, s. 3

2) $P(\underbrace{X > a}_{A}) = \int_a^{\infty} f_X(u)\,du = 1 - \underbrace{\int_{-\infty}^{a} f_X(u)\,du}_{P(A')}$

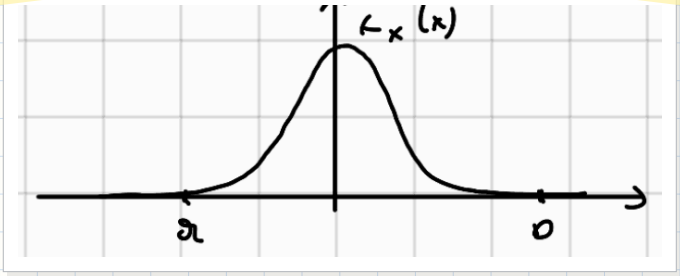
$= 1 - F_X(a)$



orijinal çizim, s. 3

3) $P(a < X \le b) = \int_a^b f_X(u)\,du = \int_{-\infty}^{b} f_X(u)\,du - \int_{-\infty}^{a} f_X(u)\,du$

$= F_X(b) - F_X(a)$

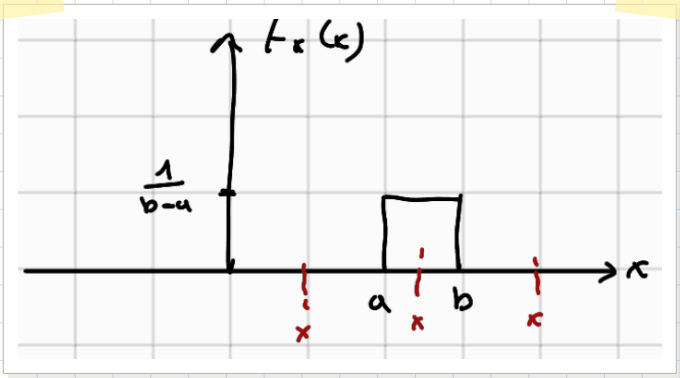


orijinal çizim, s. 3

orj. s. 4

Düzgün Dağılım (Uniform distribution)

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{diğer} \end{cases}$$



orijinal çizim, s. 4

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

1. bölge : $x < a$[1] $\Rightarrow f_X(x) = 0 \Rightarrow F_X(x) = 0$

2. bölge : $a < x < b \Rightarrow F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{a}^{x} \frac{1}{b-a}\,du$

$$= \frac{u}{b-a}\Big|_a^x$$

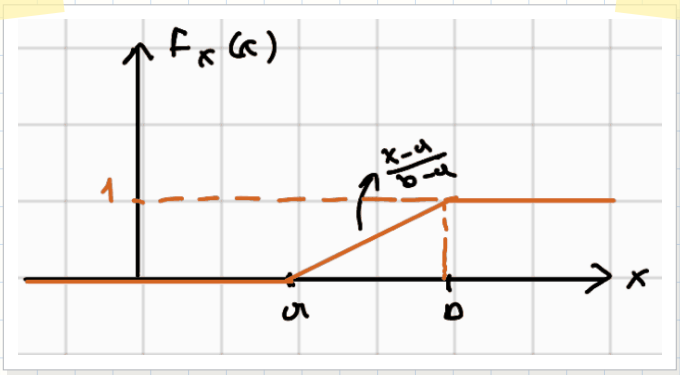
$$= \frac{x-a}{b-a}$$

[1] Claude notu: orijinalde siyah kalemle $x < 0$ yazılmış, turuncu kalemle $x < a$ olarak düzeltilmiş.

3. bölge : $x > b \Rightarrow F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_a^b \frac{1}{b-a}\,du$

$$= \frac{u}{b-a}\Big|_a^b = \frac{b-a}{b-a} = 1$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$



orijinal çizim, s. 4

Normal Dağılım (Gauss)

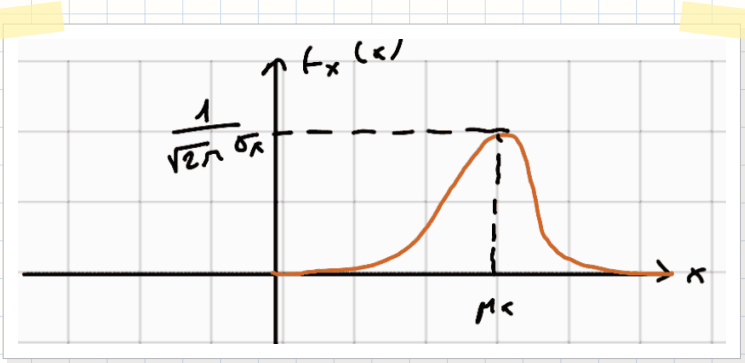
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} = N(\mu_X, \sigma_X^2)$$

μ_X : ortalama değer (mean)

σ_X : Standart sapma

σ_X^2 = Varyans

orj. s. 5

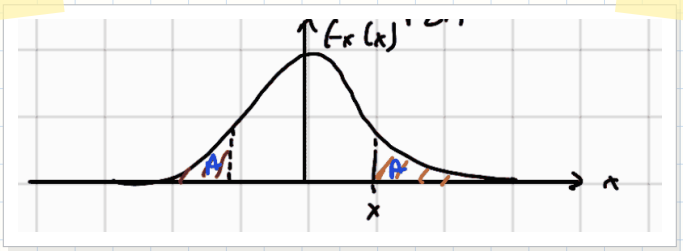


orijinal çizim, s. 5

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \frac{1}{\sqrt{2\pi}\,\sigma_X} \int_{-\infty}^{x} e^{-\frac{(u-\mu_X)^2}{2\sigma_X^2}}\,du$$

Standart Gauss Dağılımı

$$N(0,1): \quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



orijinal çizim, s. 5

$$Q(x) = P(X > x) = \int_{x}^{\infty} f_X(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}}\,dt\,, \quad x \geq 0$$

S 104 ? 5 Table B1 $x \geq 0$ — Q tablosu

$$Q(0) = 0.5$$

$$Q(-x) = 1 - Q(x)$$

$$\frac{u - \mu_X}{\sigma_X} = t$$

$$u = t\sigma_X + \mu_X \qquad du = \sigma_X\,dt$$

$$u = -\infty \rightarrow t = -\infty$$

$$u = x \rightarrow t = \frac{x - \mu_X}{\sigma_X}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \frac{1}{\sqrt{2\pi}\,\sigma_X} \int_{-\infty}^{x} e^{-\frac{(u-\mu_X)^2}{2\sigma_X^2}}\,du$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} \int_{-\infty}^{\frac{x-\mu_X}{\sigma_X}} e^{-\frac{t^2}{2}}\,\sigma_X\,dt$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \underbrace{\int_{\frac{x-\mu_X}{\sigma_X}}^{\infty} e^{-t^2/2}\,dt}_{Q\left(\frac{x-\mu_X}{\sigma_X}\right)} = 1 - Q\left(\frac{x-\mu_X}{\sigma_X}\right)$$

orj. s. 6

Beklenen Değer (Expectation)

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

↪ X'in beklenen değeri (ortalaması)

$$\underbrace{E[X^n]}_{\text{X'in n. momenti}} = \int_{-\infty}^{+\infty} x^n f_X(x)\,dx$$

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

$$g(X) = (X-\mu_X)^2 \Rightarrow \sigma_X^2 = E\left[(X-\mu_X)^2\right] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

$$= \underbrace{\int x^2 f_X(x)\,dx}_{=E[X^2]} + \mu_X^2 \underbrace{\int f_X(x)\,dx}_{1} - 2\mu_X \underbrace{\int x f_X(x)\,dx}_{\mu_X}$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = E[X^2] - (E[X])^2$$

karekteristik fonk

$$\phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)\,dx$$

$$\boxed{E[X^n] = (-j)^n \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0}}$$

$e^{\omega x}$ olduğunda moment üreten fonksiyon deniyor

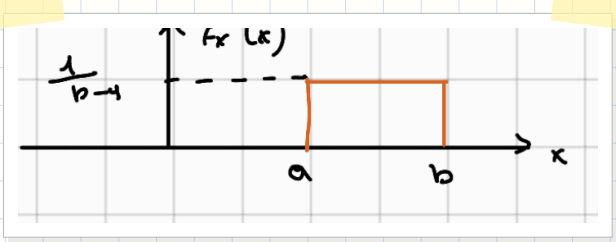
Son zamanlarda

Moment üreten fonk (MÜF):

$$M_X(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx \qquad \searrow \text{ laplace dönüşümü}$$

$$E[X^n] = \left.\frac{d^n}{ds^n} M_X(s)\right|_{s=0}$$

Örnek: Düzgün dağılım



orijinal çizim, s. 6

orj. s. 7

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$
$$= \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\frac{x^2}{2}\Big|_a^b$$
$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

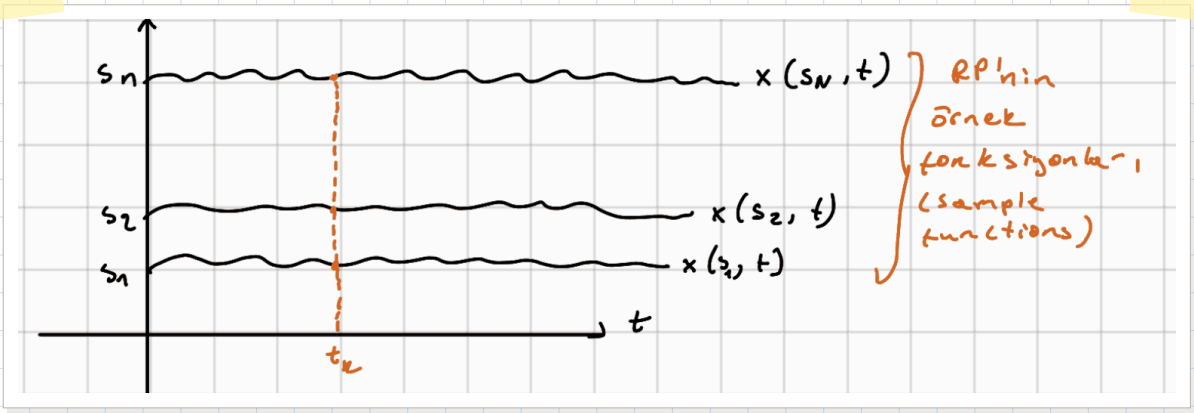
$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b$$
$$= \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$
$$= \underset{(4)}{\frac{a^2+ab+b^2}{3}} - \underset{(3)}{\frac{a^2+b^2+2ab}{4}}$$
$$= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

$$\mu_X = \frac{a+b}{2}$$
$$\sigma_X^2 = \frac{(b-a)^2}{12}$$

← önemli

Rastgele Prosesler (Random, stochastic)



orijinal çizim, s. 7

RP : $X(s,t) = X(t)$

↪ örnek fonksiyonlar topluluğu.

$X(t_k) = X_{t_k} = X_k$ RD

orj. s. 8

$$E[X_{t_k}] = \int_{-\infty}^{\infty} x_{t_k} \cdot f_{X_{t_k}}(x_{t_k})\, dx_{t_k} = \mu_{X_{t_k}}$$ RP'in beklenen değeri.

RP'in özilişki (auto correlation) fonk

$$\mathcal{R}(t_1, t_2) = E[\underbrace{X(t_1)X(t_2)}_{x_{t_1} \cdot x_{t_2}}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_{t_1} \cdot x_{t_2} \quad \underbrace{f_X(x_{t_1}, x_{t_2})}_{\substack{x_{t_1} \text{ ve } x_{t_2} \\ \text{RP'lerinin ortak OYF} \\ \text{(joint PDF)}}} \quad dx_{t_1}\, dx_{t_2}$$

Stasyonerlik

– Sıkı Stasyoner (SS) : Bütün momentler zamana bağlı değil

Strict sense stationary (SSS)

– Gevşek Stasyoner (GS) : ilk 2 moment zamana bağlı değil

Wide sense stationary (WSS)

Bir RP aşağıdaki şartları sağlıyorsa GS'dir

$$1) \quad E[X(t)] = \mu_X = \text{sabit}$$

$$2) \quad \mathcal{R}_X(t_1, t_2) = \mathcal{R}_X(\underbrace{t_1 - t_2}_{\tau}) = \mathcal{R}_X(\tau) = E[X(t)X(t-\tau)]$$

2.moment $\tau = t_1 - t_2$ ye bağlı.

Process SS $\Rightarrow$ GS

$\not\Leftarrow$

$\Leftarrow$ Gauss RP sağlar

Haberleşme sistemlerinde GS önemli:

Özellikler

- $\mathcal{R}_X(-\tau) = \mathcal{R}_X(\tau)$ çift fonk.
- $\mathcal{R}_X(0) \geq |\mathcal{R}_X(\tau)|$ ↪ complex sayılar için
- $\mathcal{R}_X(0) = E\left[|X(t)|^2\right]$ Güç

Rastgele Processler Güç işaretleridir burda istatistiksel ortalama alınıyor

orj. s. 9

- $\mathcal{R}_X(\tau) \overset{F}{\longleftrightarrow} S_X(f)$ GYF güç yoğunluk fonksiyonu

GYF

RP Güç işareti → GYF, $S_X(f)$

- $R_X(\tau) \overset{F}{\longleftrightarrow} S_X(f)$ $\quad \frac{\text{Watt}}{\text{Hertz}}$
- $S_X(f) \geq 0$ (Reel değerli)
- Prosesin ortalama gücü

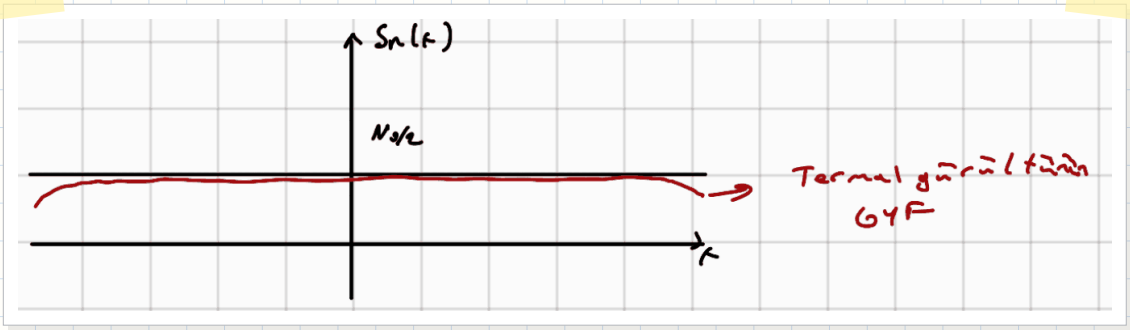
$$P_X = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\, df$$

- Reel değerli $X(t)$ RP için $S_X(-f) = S_X(f)$

Beyaz Gürültü RP

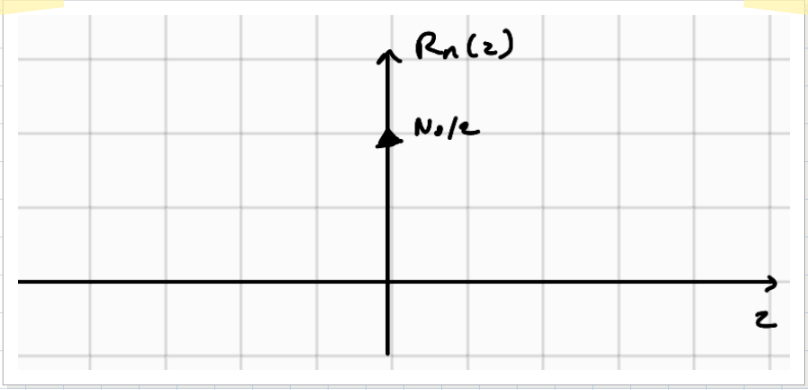
(White Noise RP)

$$n(t) \rightarrow S_n(f) = \frac{N_0}{2} = \text{sabit} \qquad \frac{\text{Watt}}{\text{Hz}} \quad \forall f$$



orijinal çizim, s. 9

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$



orijinal çizim, s. 9

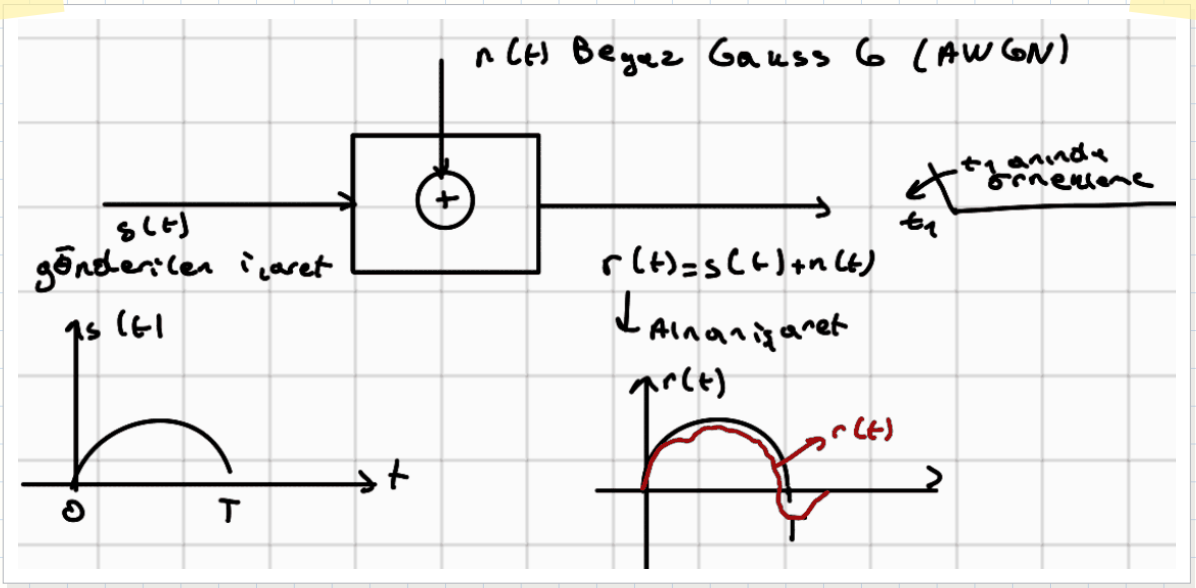
Gücü sonsuz

Gerçekte gücü sonsuz güce sahip işaret yok

Bütün frekanslarda olduğu için Beyaz gürültü denir.

(Prizma)

orj. s. 10

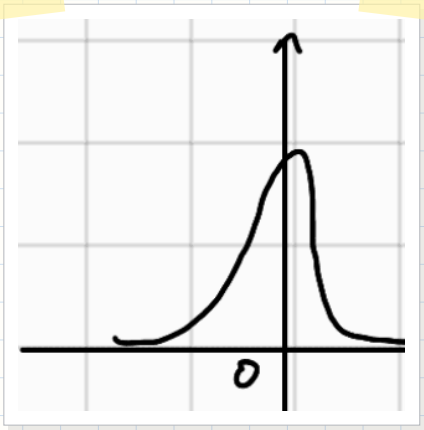


orijinal çizim, s. 10

$$\underbrace{r(t_k)}_{\substack{\text{Gauss}\\ \text{RD}}} = \underbrace{s(t_k)}_{\substack{a_i\\ \text{Deterministic}}} + \underbrace{n(t_k)}_{n} \rightarrow \text{Gauss R.D}$$

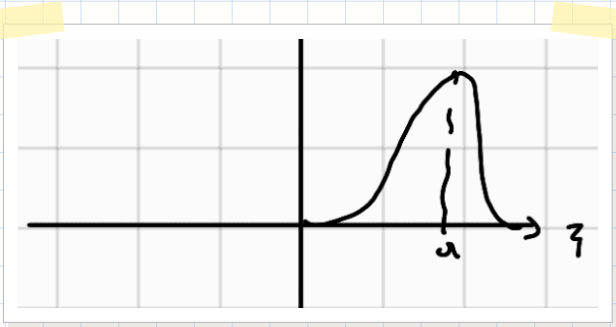
$$\underset{\substack{\downarrow\\ N(a,\sigma_n^2)}}{z} = a + n \rightarrow N(0,\sigma_n^2)$$

$$f_n(n) = \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{n^2}{2\sigma_n^2}}$$



orijinal çizim, s. 10

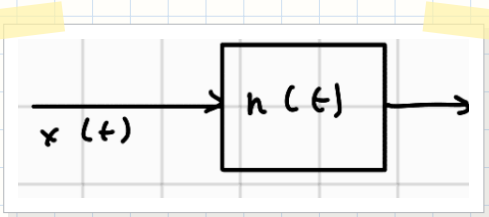
$$f_z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{(z-a)^2}{2\sigma_n^2}}$$



orijinal çizim, s. 10

Alıcı alırken z pozitif mi negatif mi diye bakıyor
s(t) pozitifken z(t)'nin negatif olma olasılığını
" negatifken " pozitif " "
bulup toplayıp ikiye bölerek yapıyor

LZD sistemler



orijinal çizim, s. 10

$$y(t) = x(t) \otimes h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$X(t)$ RP — $Y(t)$ RP

$S_X(f)$ — $S_Y(f) = S_X(f)\,|H(f)|^2$

İşaretler ve Sistemler

$(-T/2,\ T/2)$ arasında harcanan enerji

$$E_X^T = \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

orj. s. 11

güç $P_X^T = \frac{E_X^T}{T}$

$$E_X = \lim_{T\to\infty} E_X^T = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$P_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$E_X < \infty \Rightarrow x(t)$ enerji işareti
$P_X < \infty \Rightarrow$ " güç işareti

$$x(t) \text{ periyodik } \Rightarrow P_X = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

$$x(t) = \begin{cases} A\cos(2\pi f_0 t) \\ A\sin(2\pi f_0 t) \end{cases} \Rightarrow P_X = \frac{A^2}{2} \text{ watt}$$