Sayısal Haberleşme

Temel işaretler, Fourier dönüşümü, örnekleme

Orijinal not: Ders3.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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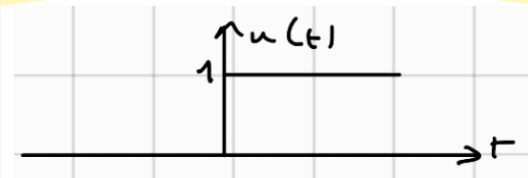
orj. s. 1

Bazı önemli işaretler

Unit step signal

(Birim basamak işareti)

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

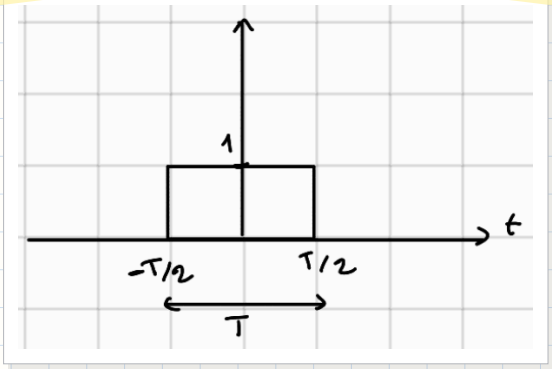


orijinal çizim, s. 1

Rectangular pulse signal

Dikdörtgen darbe işareti

$$\Pi\left(\frac{t}{T}\right) = \text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{diğer} \end{cases}$$

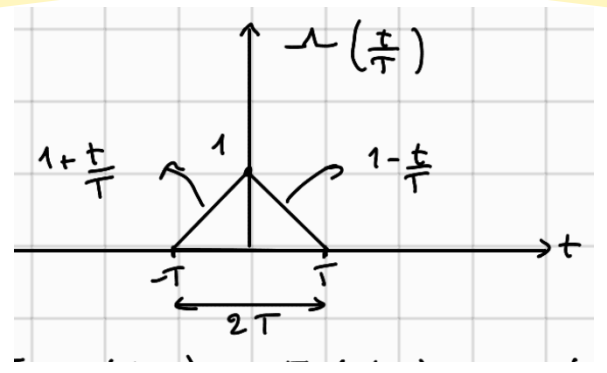


orijinal çizim, s. 1

Triangular pulse signal

Üçgen darbe işareti

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T} & |t| \le T \\ 0 & \text{diğer} \end{cases}$$



orijinal çizim, s. 1

$$T\Lambda\left(\frac{t}{T}\right) = \Pi\left(\frac{t}{T}\right) \circledast \Pi\left(\frac{t}{T}\right)$$

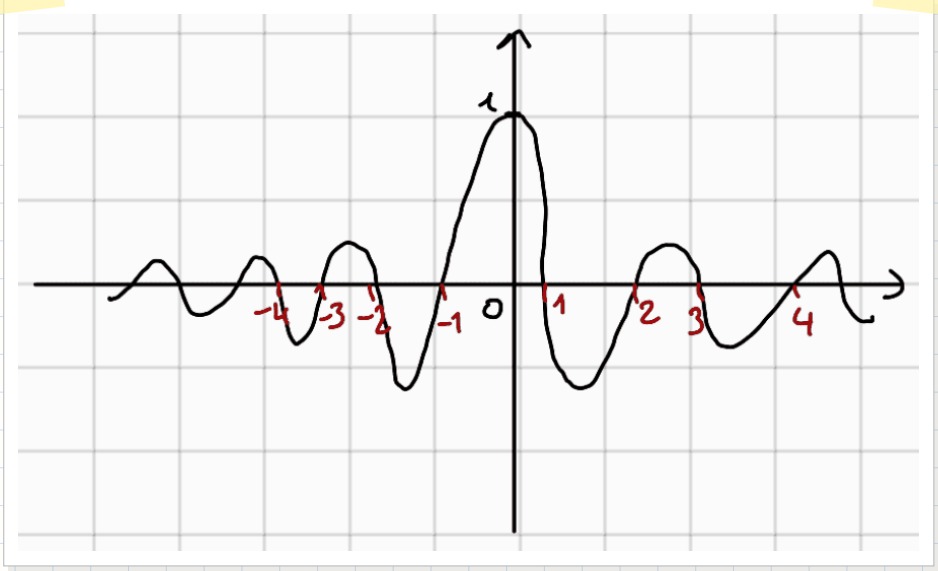
$$\Lambda(t) = \Pi(t) \circledast \Pi(t)$$

orj. s. 2

Sinc pulse signal

(sinc darbe işareti)

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

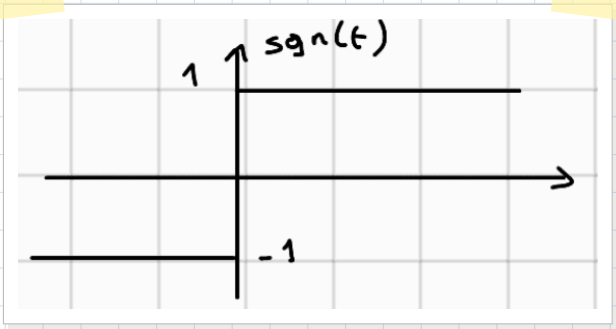


orijinal çizim, s. 2

Sgn signal

(işaret fonksiyonu)

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

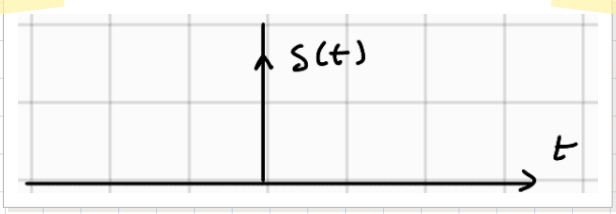


orijinal çizim, s. 2

Unit impulse (Dirac delta function)

Birim darbe fonksiyonu

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{diğer} \end{cases}$$



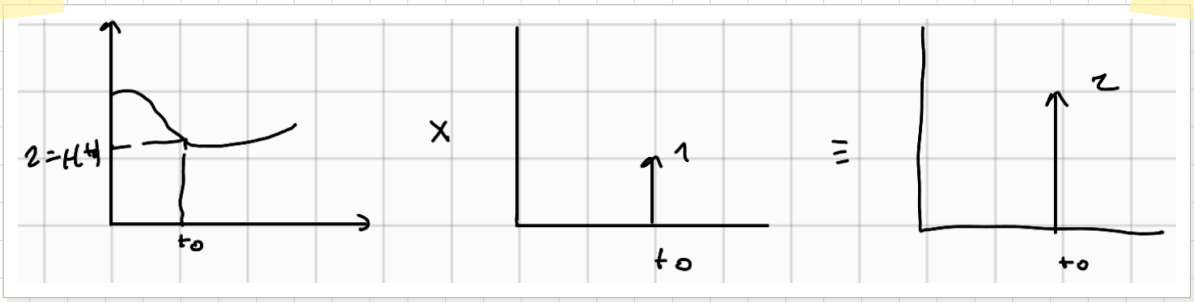
orijinal çizim, s. 2

Özellikler:

1) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

2) Sifting / sampling property

$$f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0), \qquad t_0 = \text{sbt}$$



orijinal çizim, s. 2

orj. s. 3

$$\int_a^b \underbrace{f(t)\,\delta(t-t_0)}_{f(t_0)\,\delta(t-t_0)}\,dt = \begin{cases} f(t_0), & a < t_0 < b \\ 0, & \text{diğer} \end{cases}$$

$$\int_a^b \delta(t-t_0)\,dt = \begin{cases} 1, & a < t_0 < b \\ 0, & \text{diğer} \end{cases}$$

3) $\forall a \neq 0 \qquad \delta(at) = \frac{1}{|a|}\,\delta(t)$

$$a = -1 \Rightarrow \delta(-t) = \delta(t) \quad \text{çift fonk.}$$

4) $f(t) \circledast \delta(t) = f(t)$
$f(t) \circledast \delta(t-t_0) = f(t-t_0)$

5) $v(t) = \int_{-\infty}^{t} \delta(\alpha)\,d\alpha \Rightarrow \delta(t) = \frac{dv(t)}{dt}$

Fourier series

$$\forall t \quad x(t+T_0) = x(t)$$

$$\underbrace{x(t)}_{\text{FS açılımı}} = \sum_{n=-\infty}^{+\infty} c_n\, e^{j2\pi n f_0 t}$$

$$\underbrace{c_n}_{\text{FS katsayıları}} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,e^{-j2\pi n f_0 t}\,dt$$

$$\int_0^{T_0} e^{j2\pi n f_0 t}\,e^{-j2\pi m f_0 t}\,dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases}$$

$$c_n = |c_n|\,e^{j\theta_n}$$

$$c_{-n} = |c_n|\,e^{-j\theta_n}$$

Parseval teoremi

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Örnek

$$x(t) = 2\sin(100t)$$

$$\omega_0 = 100 = \frac{2\pi}{T_0} \rightarrow T_0 = \frac{\pi}{50}\ \text{sn}$$

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt = \frac{1}{\pi/50}\int_0^{\pi/50} 4\ \underbrace{\sin^2(100t)}_{\frac{1}{2}[1-\cos(200t)]}\ dt$$

◯ 0: Frekansı iki katına çıkmış, periyodu $T_0/2$ onun için

$$= \frac{1}{\pi/50}\,2\underbrace{\int_0^{\pi/50} dt}_{\pi/50} = 2\ \text{watt}$$

$$P_x = \frac{A^2}{2} = \frac{4}{2} = 2\ \text{watt}$$

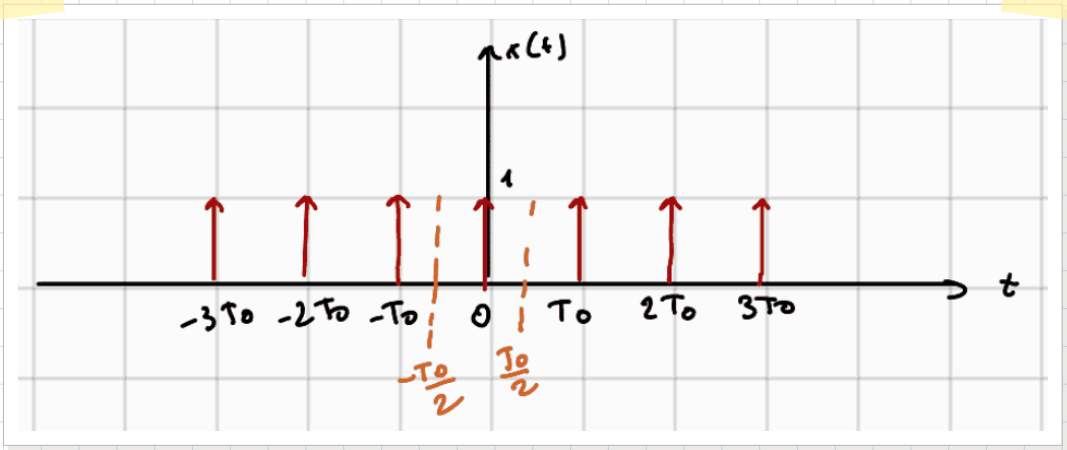
orj. s. 4

$$x(t) = 2\sin(100t) = 2\left(\frac{e^{j100t} - e^{-j100t}}{2j}\right) = \underbrace{\frac{1}{j}}_{c_1} e^{j100t}\ \underbrace{-\frac{1}{j}}_{c_{-1}} e^{-j100t}$$

$$P_x = \sum_n |c_n|^2 = \left|\frac{1}{j}\right|^2 + \left|-\frac{1}{j}\right|^2 = 2\ \text{watt}$$

Örnek: Dürtü katarı

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$



orijinal çizim, s. 4

$$c_n = \frac{1}{T_0}\underbrace{\int_{-T_0/2}^{T_0/2} \underbrace{x(t)}_{\delta(t-0)}\, e^{-j2\pi n f_0 t}\, dt}_{e^{-j2\pi n f_0 \cdot 0}=1} = \frac{1}{T_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} e^{j2\pi n f_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{j2\pi n f_0 t}$$

Fourier Dönüşümü

– Periyodik olmayan işaretler için

Dirichlet şartları

1) $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$
2) Reel eksendeki herhangi bir sonlu aralıkta sonlu sayıda maksimum ve minimum olmalıdır.
   3- (süreksizliği)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, dt^{1}$$

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$

$$X(f) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\}$$

orj. s. 5

$$X(f) = \underbrace{|X(f)|}_{\text{Genlik spektrumu}} \quad \underbrace{e^{j\theta_X(f)}}_{\text{Faz spektrumu}}$$

$$x(t) \text{ reel} \Rightarrow X(-f) = X^*(f)$$

$$x(t) \text{ reel ve çift} \Rightarrow X(f) \underbrace{\text{reel}}_{X^*(f)=X(f)} \text{ ve } \underbrace{\text{çift}}_{X(-f)=X(f)}$$

$$x(t) \text{ reel ve tek} \Rightarrow X(f) \underbrace{\text{sanal}}_{X^*(f)=-X(f)} \text{ ve } \underbrace{\text{tek}}_{X(-f)=-X(f)}$$

Lineerlik: $\underbrace{a x_1(t) + b x_2(t)}_{x_3(t)} \overset{F}{\longleftrightarrow} \underbrace{a X_1(f) + b X_2(f)}_{X_3(f)}$

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f \cdot 0} = 1$$

(t göründüğü yere 0 koy)

$$\delta(t) \overset{F}{\longleftrightarrow} 1$$

[1] Claude notu: muhtemelen df olmalı.

$$\mathcal{F}^{-1}\{\delta(f)\} = \int_{-\infty}^{\infty} \delta(f)\, e^{j2\pi ft}\, df = e^{j2\pi \cdot 0 \cdot t} = 1$$

$$\delta(f) \overset{\mathcal{F}^{-1}}{\longleftrightarrow} 1$$

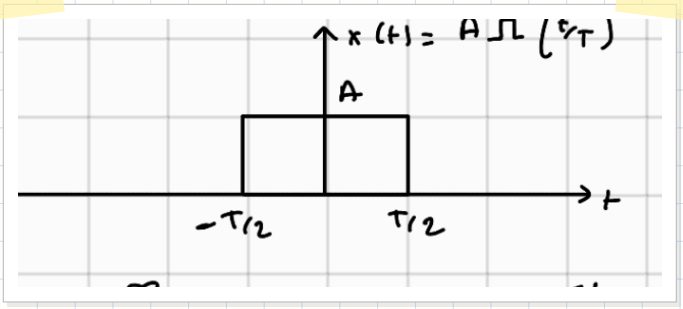
$$1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(f)$$

$$\mathrm{Sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\pi f}$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

$$\mathcal{F}\{u(t)\} = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

Örnek



orijinal çizim, s. 5

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = A\int_{-T/2}^{T/2} e^{-j2\pi ft}\, dt = A\,\frac{e^{-j2\pi ft}}{-j2\pi f}\Bigg|_{-T/2}^{T/2}$$

$$= \frac{A}{\pi f}\,\underbrace{\frac{e^{-j\pi fT} - e^{j\pi fT}}{-2j}}_{\sin(\pi fT)}$$

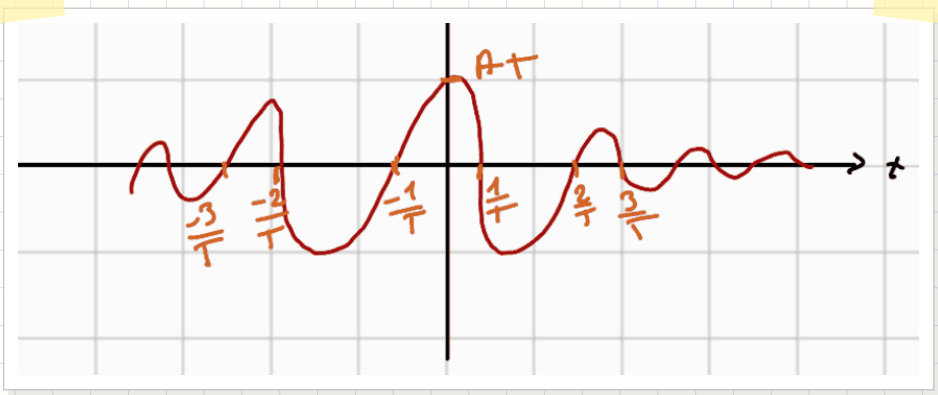
$$X(f) = AT\,\frac{\sin(\pi fT)}{\pi fT} = AT\,\mathrm{sinc}(fT)$$

orj. s. 6

$$x(t) = A\,\Pi\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = AT\,\mathrm{sinc}(fT)$$

$$\pi fT = \pi k, \quad k \in \mathbb{Z} - \{0\}$$

$$f = \frac{k}{T}$$



orijinal çizim, s. 6

$$BG = \frac{1}{T}\ \text{Hz}$$

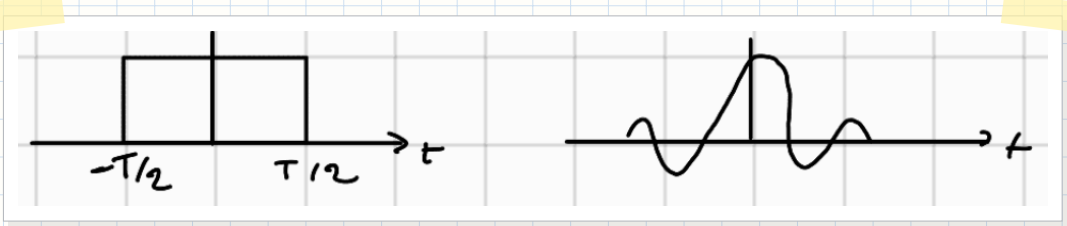
zamanda		Frekans uzayında
T	x	$BG = 1/T$

Dualite özelliği

$$x(t) \overset{F}{\longleftrightarrow} X(f)$$

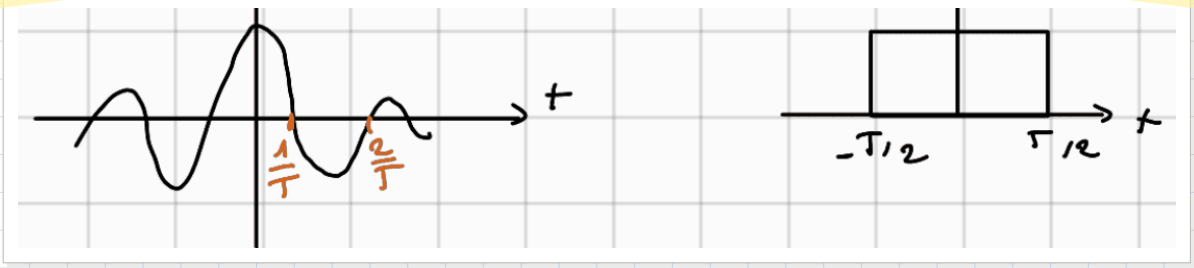
$$y(t) = X(t) \longleftrightarrow Y(f) = x(-f)$$

$$x(t) = A\,\Pi\left(\frac{t}{T}\right) \longleftrightarrow X(f) = AT\,\text{sinc}(fT)$$



orijinal çizim, s. 6

$$y(t) = X(t) = A\,\text{sinc}(tT)^2 \longleftrightarrow Y(f) = x(-f) = A\,\Pi\left(\frac{-f}{T}\right) = A\,\Pi\left(\frac{f}{T}\right)$$



orijinal çizim, s. 6

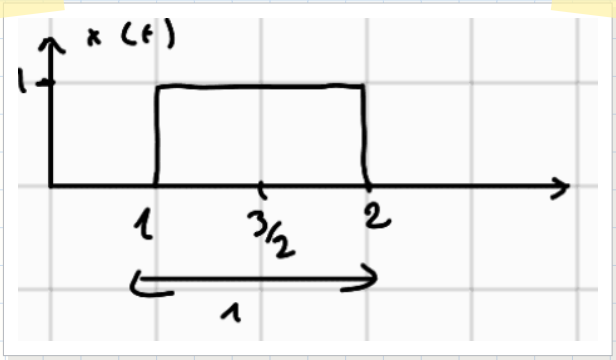
[2] Claude notu: muhtemelen $AT\,\text{sinc}(tT)$ olmalı.

$$AT\,\mathrm{sinc}(fT) \overset{F}{\longleftrightarrow} A\,\Pi\left(\frac{f}{T}\right)$$

Zamanda öteleme özelliği

$$x(t) \longleftrightarrow X(f)$$
$$x(t-t_0) \longleftrightarrow X(f)\,e^{-j2\pi f t_0}$$

ikisinin genlik spektrumu aynı, bant genişliği de aynıdır.



orijinal çizim, s. 6

$$x(t) = \Pi\left(t-\frac{3}{2}\right)$$
$$\downarrow$$
$$X(f) = \mathrm{sinc}(f)\,e^{-j2\pi f\frac{3}{2}}$$

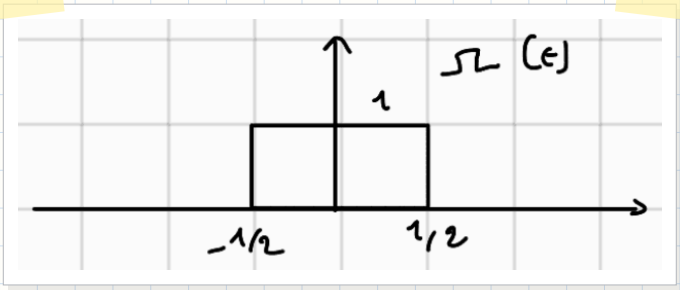
orj. s. 7

Ölçekleme özelliği

(scaling property)

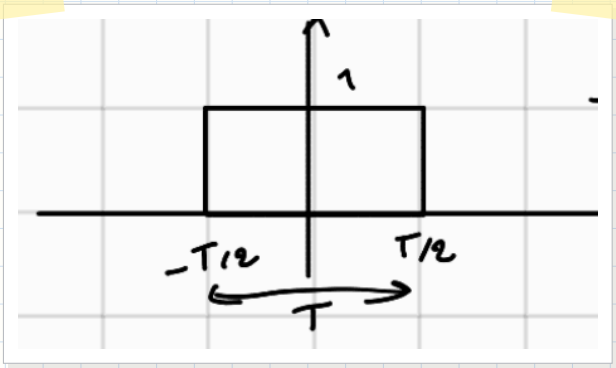
$$a \neq 0,\ a \in \mathbb{R}$$
$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

	zaman		frekans
$a > 1 \Rightarrow$	$x(at)$ Daralma	$\Rightarrow$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$ genişleme
$a < 1 \Rightarrow$	$x(at)$ genişleme	$\Rightarrow$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$ Daralma olacak



orijinal çizim, s. 7

$$\xrightarrow{F} \operatorname{sinc}(f)$$



orijinal çizim, s. 7

$$\Pi\left(\frac{t}{T}\right) \xrightarrow[a=1/T]{F} T\operatorname{sinc}(fT)$$

Konvolüsyon özelliği

$$x(t) \circledast y(t) = \int_{-\infty}^{\infty} x(z)\,y(t-z)\,dz = \int_{-\infty}^{\infty} y(z)\,x(t-z)\,dz$$

$$z(t) = x(t) \circledast y(t) \xleftrightarrow{F} X(f)\,Y(f) = Z(f)$$

$$T\Lambda\left(\frac{t}{T}\right) = \Pi\left(\frac{t}{T}\right) \circledast \Pi\left(\frac{t}{T}\right)$$

$$\downarrow$$

$$T\mathcal{F}\left\{\Lambda\left(\frac{t}{T}\right)\right\} = T\operatorname{sinc}(fT)\cdot T\operatorname{sinc}(fT)$$

$$\boxed{\Lambda\left(\frac{t}{T}\right) \xleftrightarrow{F} T\operatorname{sinc}^2(fT)}$$

Frekansta öteleme ve modülasyon özelliği

$$x(t) \xleftrightarrow{F} X(f)$$

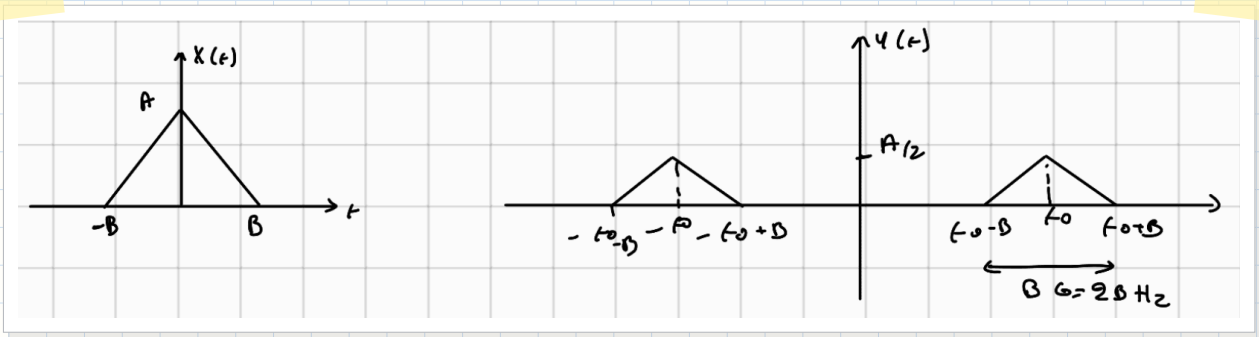
$$x(t)\,e^{j2\pi f_0 t} \xleftrightarrow{F} X(f-f_0)$$

Örnek:

$$g(t) = x(t)\cos(2\pi f_0 t) = x(t)\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

$$\overset{F}{\longleftrightarrow} Y(f) = \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$$

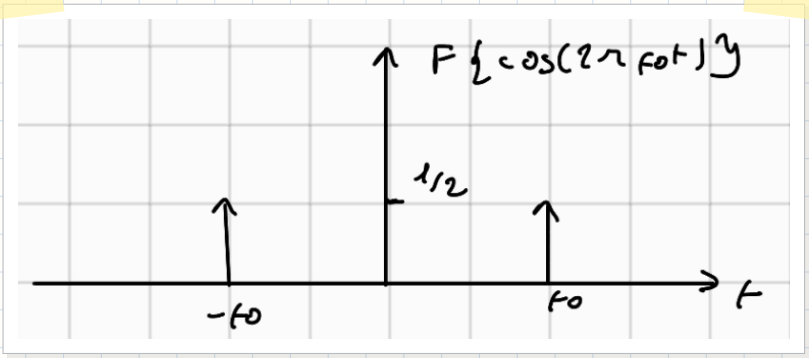
orj. s. 8



orijinal çizim, s. 8

$$x(t) = 1 \Rightarrow e^{j2\pi f_0 t} \overset{F}{\longleftrightarrow} \delta(f - f_0)$$

$$\cos(2\pi f_0 t) \overset{F}{\longleftrightarrow} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$



orijinal çizim, s. 8

$$\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} \overset{F}{\longleftrightarrow} \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

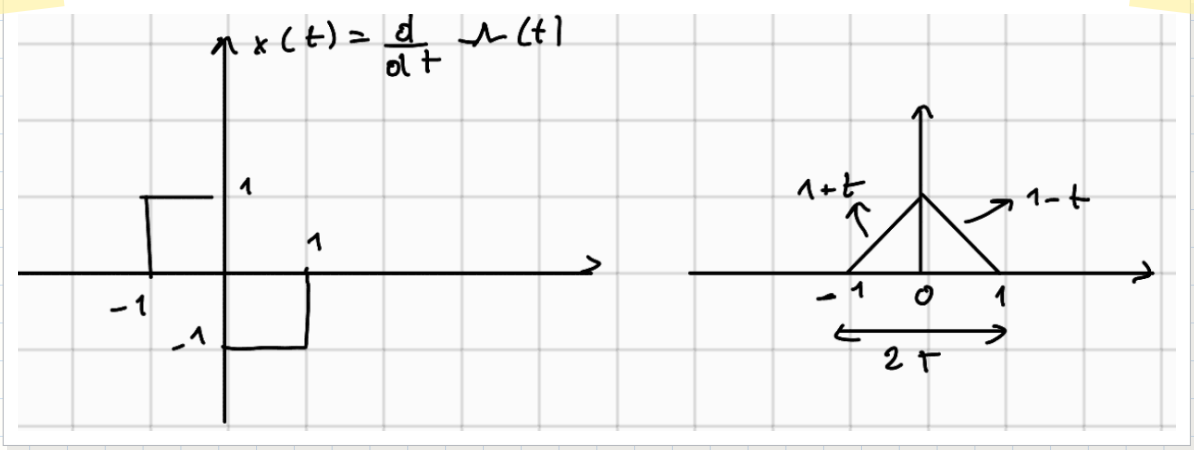
Zamanda türev

$$\frac{d}{dt}x(t) \overset{F}{\longleftrightarrow} j2\pi f X(f)$$

yüksek frekansları kuvvetlendirir (Basit bir yüksekgeçiren filtre)

$$\frac{d^n}{dt^n}x(t) \overset{F}{\longleftrightarrow} (j2\pi f)^n X(f)$$

Örnek:



orijinal çizim, s. 8

$$X(f) = j2\pi f \underbrace{\mathcal{F}\{\Lambda(t)\}}_{\text{sinc}^2(f)} = j2\pi f \,\text{sinc}^2(f)$$

orj. s. 9

Frekansta türev

$$t\,x(t) \longleftrightarrow \frac{j}{2\pi}\frac{d}{df}X(f)$$

$$t^n x(t) \longleftrightarrow \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n}X(f)$$

İntegral özelliği:

$$y(t) = \int_{-\infty}^{t} x(z)\,dz \overset{F}{\longleftrightarrow} Y(f) = \frac{X(f)}{j2\pi f} + \frac{1}{2}\overbrace{X(0)}^{X(f)|_{f=0}\text{ daki değer}}\delta(f)$$

Frekansta konvolüsyon özelliği

$$\underbrace{\underbrace{x_1(t)}_{B_1\text{ bant genişliği}} \cdot \underbrace{x_2(t)}_{B_2\text{ bant genişliği}}}_{z(t)} \overset{F}{\longleftrightarrow} \underbrace{X_1(f) \circledast X_2(f)}_{Z(f)}$$

$Z(f)$'in bant genişliği $B_1 + B_2$

Periyodik işaretlerin Fourier Dönüşümü

$$\forall t \quad x(t + T_0) = x(t)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t}, \qquad c_n = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-j2\pi n f_0 t}\,dt$$

$$e^{j2\pi nf_0 t} \overset{F}{\longleftrightarrow} \delta(f - nf_0)$$

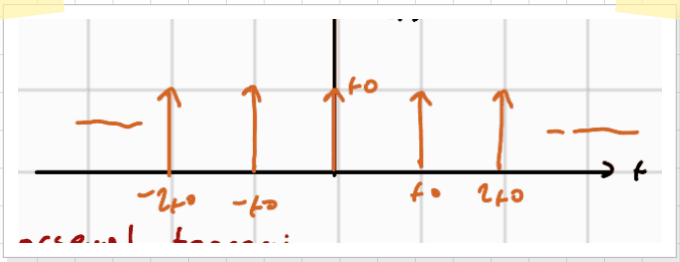
$$e^{j2\pi nf_0 t} \overset{F}{\longleftrightarrow} \delta(f - nf_0)$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nf_0 t} \overset{F}{\longleftrightarrow} \boxed{X(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_0)}$$

Dürtü katarı:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{j2\pi nf_0 t}$$

$$\overset{F}{\longleftrightarrow} X(f) = \overbrace{\frac{1}{T_0}}^{f_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

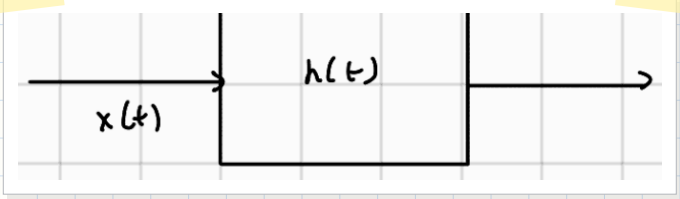


orijinal çizim, s. 9

Parseval teoremi

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} \underbrace{|X(f)|^2}_{\varepsilon_x(f)} \, df$$

$\varepsilon_x(f)$: Enerji yoğunluk fonk.



orijinal çizim, s. 9

$$y(t) = x(t) \circledast h(t)$$

$Y(f) = X(f) \cdot H(f) \rightarrow$ Sistemin transfer fonksiyonu

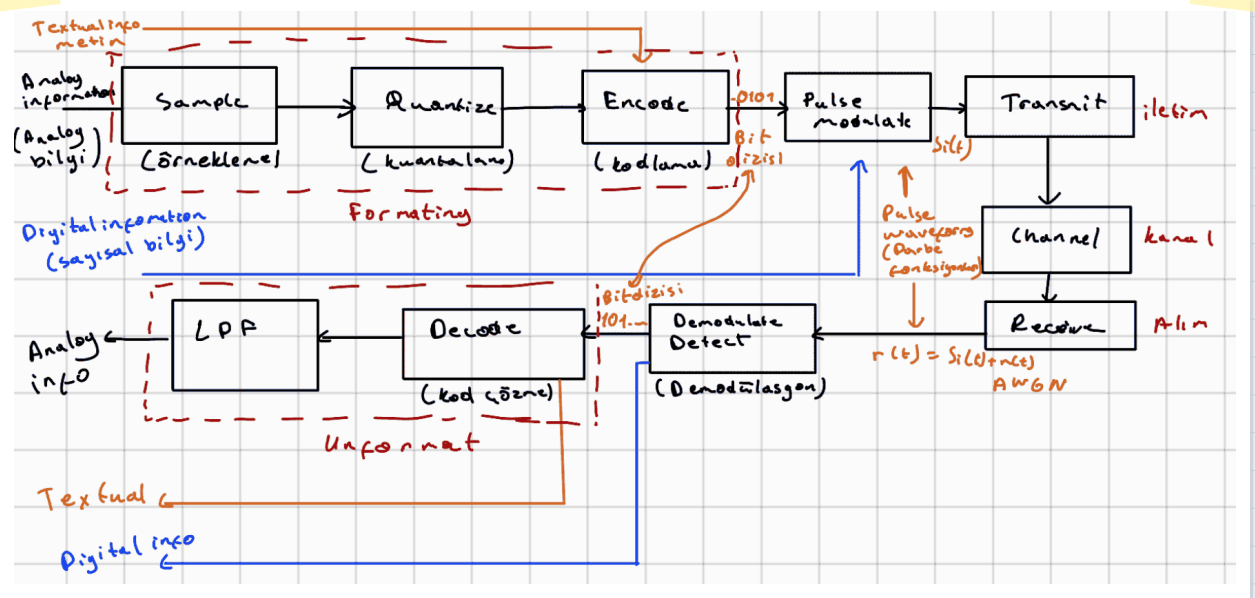
orj. s. 10

RP	RP
$S_x(f)$	$S_y(f) = S_x(f)\,\lvert H(f)\rvert^2$
GYF	GYF

Asıl derse geçelim

Kaynak kodlaması (Source coding)

Ch2: Formatting and Baseband communication



orijinal çizim, s. 10

Örnekleme Teoremi (Sampling Theorem)

A bandlimited signal having no spectral components above f_m Hz can be uniquely determined by values sampled at uniform intervals of $T_s \leq \frac{1}{2f_m}$ sec.

$\longrightarrow$ In order to reconstruct a signal with a BW of f_m completely from its samples, the sampling rate $f_s \geq 2f_m$.

En yüksek frekansı f_m Hz olan bir işareti düzgün aralıklarla alınan örneklerinden yeniden elde etmek için örnekleme hızı $f_s \geq 2f_m$ olmalıdır.

$$f_s = \frac{1}{T_s} \geq 2f_m$$

f_s: Örnekleme frekansı, (sampling frequency)
T_s: Örnekleme aralığı (Sampling interval (period))

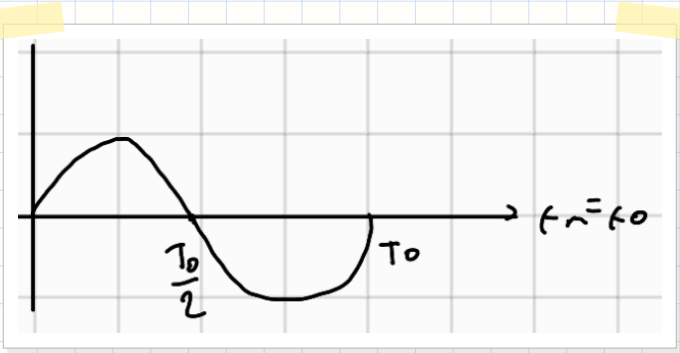
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$$\underbrace{f_N}_{\text{Nyquist frekansı (rate)}} = 2f_m \longrightarrow f_s \geq f_N$$

? Δ saf sinüsoidal işaret her için

$$f_s = f_N + \delta$$

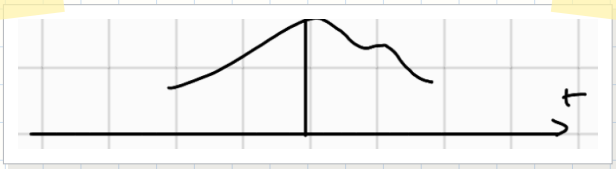
$\delta \ll$ olmalıdır (en az)



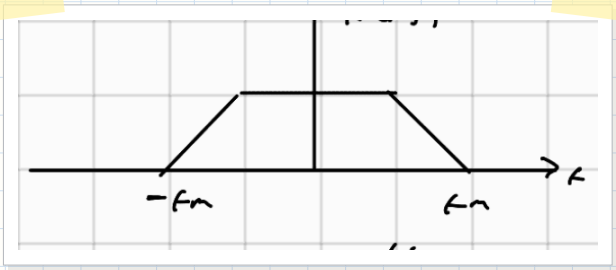
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$f_s = 2f_m$ olursa $T_0/2$ de örnek alınır örnekler 0 olur

İdeal örnekleme (Ideal Sampling / Impulse sampling)



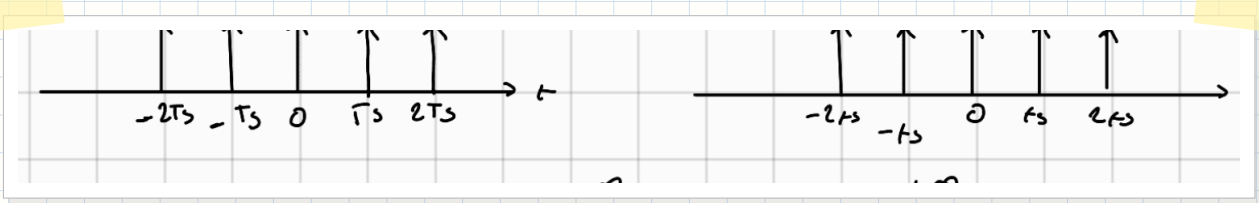
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Dürtü katarı

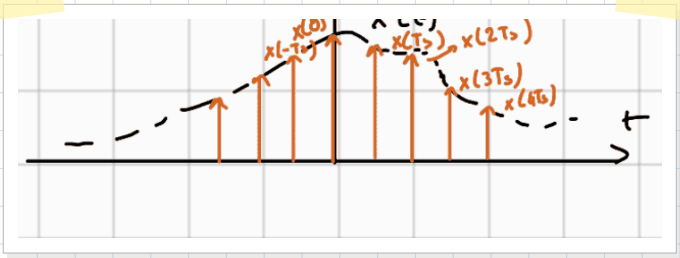
$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \overset{F}{\longleftrightarrow} X_\delta(f) = \overbrace{\frac{1}{T_s}}^{=f_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$



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$$\underbrace{x_s(t)}_{\substack{\text{Örneklenmiş işaret}\\ \text{(Dürtü katarlı)}}} = x(t)\cdot x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

$$= \underbrace{\sum_{n=-\infty}^{+\infty} x(nT_s)\,\delta(t-nT_s)}_{f(t)\,\delta(t-t_0)=f(t_0)\,\delta(t-t_0)}$$



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$$x_s(t) = x(t)\cdot x_\delta(t) \overset{F}{\longleftrightarrow} X_s(f) = X(f)\circledast X_\delta(f)$$

$$= X(f)\circledast f_s\sum_{n=-\infty}^{+\infty}\delta(f-nf_s)$$

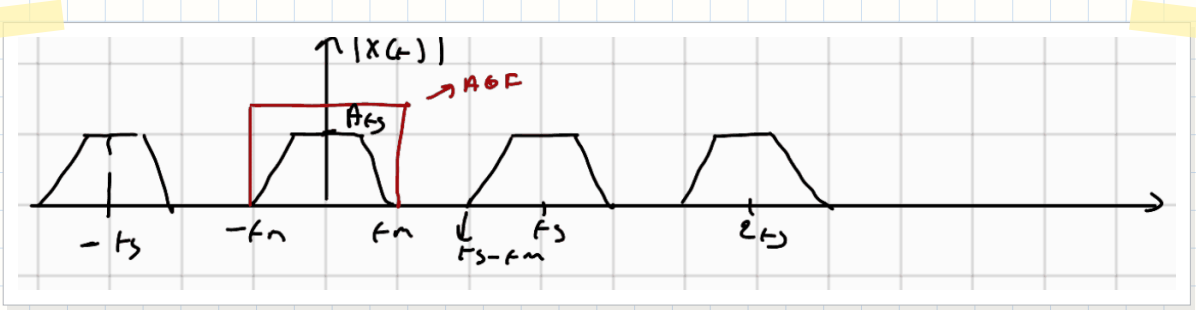
$$= f_s\sum_{n=-\infty}^{\infty}\underbrace{[X(f)\circledast\delta(f-nf_s)]}_{X(f-nf_s)}$$

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$$(f(t)\circledast\delta(t-t_0) = f(t-t_0))$$

$$X_s(f) = f_s\sum_{n=-\infty}^{\infty}X(f-nf_s)$$

$$X_s(f) = f_s\sum_{n=-\infty}^{\infty}X(f-nf_s)$$



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$$f_s - f_m > f_m$$

$$\Rightarrow f_s \geq 2f_m$$