Sayısal İşaret İşleme

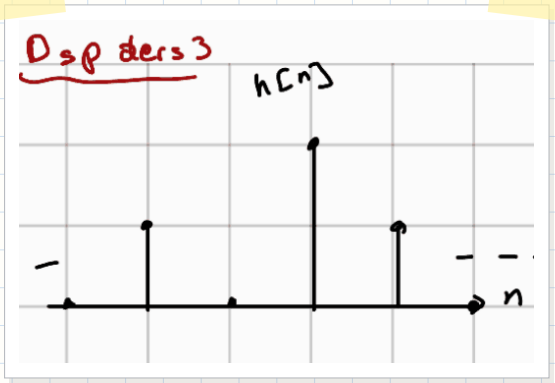
Fark denklemleri, FIR/IIR, DTFT ve frekans cevabı

Orijinal not: Dsp-ders-3.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

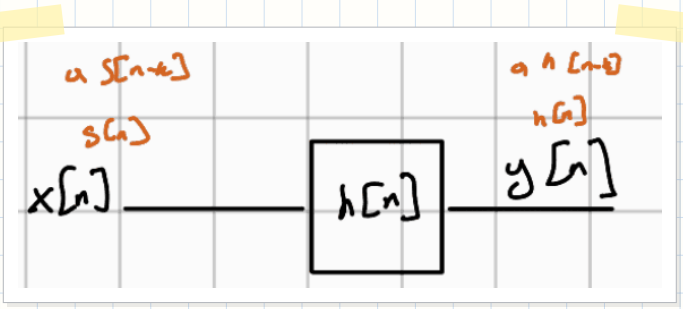
NOTICE *This document was transcribed from handwritten lecture notes with the help of Claude (Anthropic). It is meant as a supplementary resource and may contain errors or omissions. Always check against the original notes and official course material.*

orj. s. 1



orijinal çizim, s. 1

FIR : sonlu sayıda 0'dan farklı değer alır



orijinal çizim, s. 1

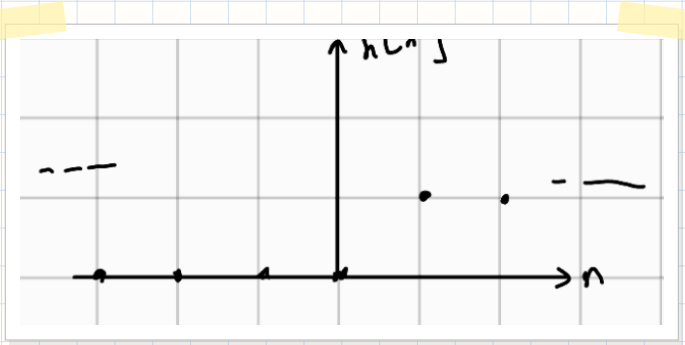
ve $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

$$y[n] = x[n] * h[n]$$

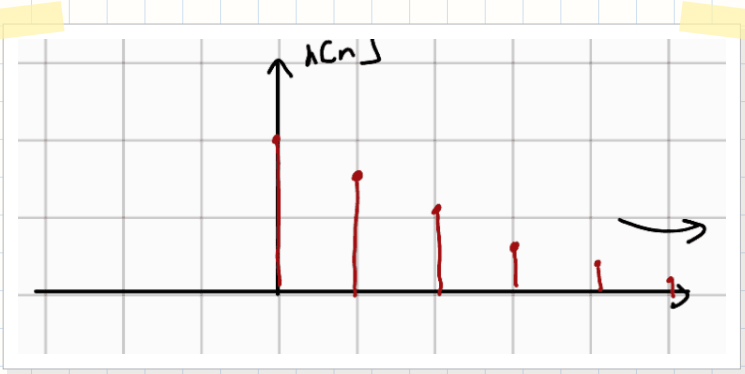
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

↓ ? bunse ↙ ? bu dummy

$$= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$



orijinal çizim, s. 1



orijinal çizim, s. 1

→ olsa bir yerden sonra toplamı kesebiliriz

Filtre IIR ise konvülasyon sadece kağıt üzerinde gerçekleşebilir
yani toplam formülü kullanılamaz.

$$y[n] + \sum_{k=1}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$y[n] = \sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{N} a_k\, y[n-k]$$

Constant coefficient Difference Equation

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_M x[n-M] - a_1 y[n-1] - a_2 y[n-2]$$

Sabit katsayılı fark denklemi

Bu nereden geliyor??

Bir sistem düşünelim[1]

$$y[n] = \tfrac{1}{3}x[n] + \tfrac{1}{3}x[n-1] + \tfrac{1}{3}x[n-2]$$

$$\longrightarrow\ h[n] = \tfrac{1}{3}\delta[n] + \tfrac{1}{3}\delta[n-1] + \tfrac{1}{3}\delta[n-2]$$

$$y[n-1] = \tfrac{1}{3}x[n-1] + \tfrac{1}{3}x[n-2] + \tfrac{1}{3}x[n-3]$$

[1] Orijinalde turuncu kalemle üstü çizilen terimler turuncu vurguyla gösterilmiştir.

$$y[n] - y[n-1] = \frac{1}{3}x[n] - \frac{1}{3}x[n-3]$$
$$y[n] = \frac{1}{3}x[n] - \frac{1}{3}x[n-3] + y[n-1]$$

$b_0 = \frac{1}{3}$
$b_1 = 0$
$b_2 = 0$
$b_3 = -\frac{1}{3}$
$b_4 = 0$

$a_1 = 1$[2]
$a_2 = 0$
$a_3 = 0$

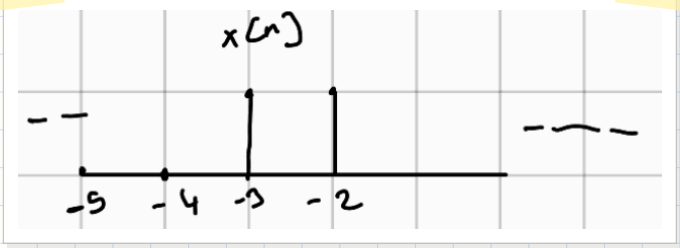
orj. s. 2

Örnek

$$y[n] = \frac{1}{100}x[n] + \frac{1}{100}x[n-1] + \cdots + \frac{1}{100}x[n-99]$$

$$y[n] - y[n-1] = \frac{1}{100}x[n] - \frac{1}{100}x[n-100]$$
$$y[n] = \frac{1}{100}x[n] - \frac{1}{100}x[n-100] + y[n-1]$$

$y[n] = 5x[n] + 3 \quad \Rightarrow$ lineer değil, LTI değil



orijinal çizim, s. 2

Başlangıç koşulları

$y[-4] = 0$
$y[-5] = 0$
$y[-6] = 0$
$y[-1] = y[0] = 0$

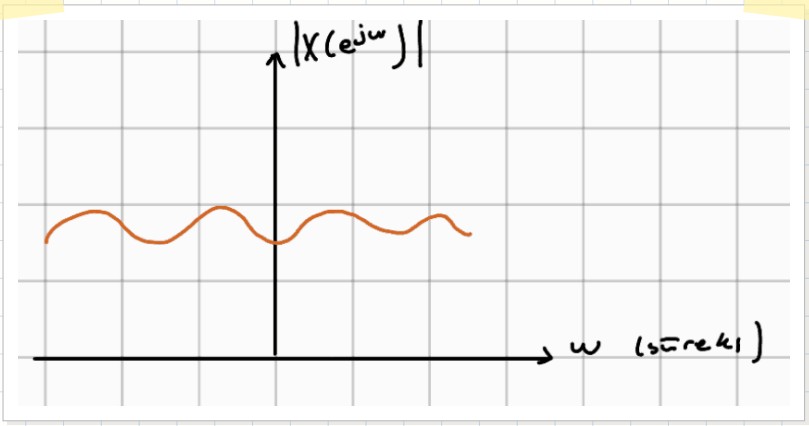
DTFT

→ sonlu olmalı

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

[2] Claude notu: yukarıdaki fark denklemine göre muhtemelen $a_1 = -1$ olmalı.

ω'nın fonksiyonu sürekli



orijinal çizim, s. 2

$$\cos(\Omega t)\Big|_{t=nT}$$
$$t = \frac{n}{F_s}$$
$$\cos\left(\frac{\Omega}{F_s} n\right)$$
$$\omega = \frac{\Omega}{F_s}$$
$$\cos(\omega n)$$
$$-\infty < \Omega < \infty$$
$$-\infty < \omega < \infty$$

$X(e^{j\omega})$ ω'nın 2π'sinde periyodik

$$0 \le \omega < 2\pi$$
$$-\pi \le \omega < \pi$$

(ses işareti 3,2 | 4 kHz'e sınırlanıyor
8 kHz örnekleme frekansı)

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$
$$= |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$$

↗ Fourier için yeter koşuldur

$$|X(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right| \le \sum_{k=-\infty}^{\infty} |x[k]\, e^{-j\omega k}|$$
$$\le \sum_{k=-\infty}^{\infty} |x[k]| \le \infty$$

orj. s. 3

↓ bu şartı sağlamayan
FIR sistemlerin her zaman fourieri
IIR sistemlerde sadece kararlıysa var

Properties of DTFT

$$X(e^{j\omega}) = X(e^{j\omega}e^{j2\pi}) = X(e^{j(\omega+2\pi)}) \qquad \text{Periyodik } 2\pi \text{ ile}$$

$$x_1[n] \longleftrightarrow X_1(e^{j\omega})$$
$$x_2[n] \longleftrightarrow X_2(e^{j\omega})$$
$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$
$$\cos(\omega n) = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\omega n}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$
$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}X(e^{j\omega})$$
$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

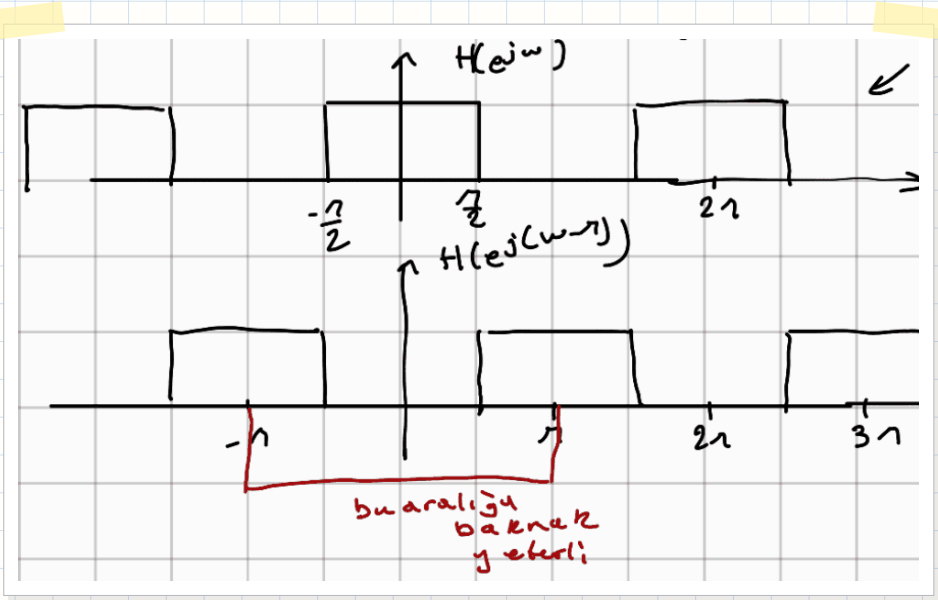
ispat

$$\begin{cases} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n} \\ X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\,x[n]\,e^{-j\omega n} \\ \qquad\quad = \sum_{n=-\infty}^{\infty} x[n]\,e^{j(\omega-\omega_0)n}{}^{3} \end{cases}$$

$$\omega_0 = \pi \qquad e^{j\pi n} = (-1)^n$$

$$h[n] \longleftrightarrow H(e^{j\omega})$$
$$(-1)^n h[n] \longleftrightarrow H(e^{j(\omega-\pi)})$$

[3] Claude notu: muhtemelen $e^{-j(\omega-\omega_0)n}$ olmalı.



orijinal çizim, s. 3

alçak geçiren

yüksek geçiren çünkü en yüksek frekans π

$$x[n] \longleftrightarrow X(e^{j\omega})$$
$$x^*[n] \longleftrightarrow X^*(e^{-j\omega}) \quad \leftarrow \text{konjüge simetrik}$$

$$x[n] \text{ reel} \Rightarrow x[n] = x^*[n]$$
$$X(e^{j\omega}) = X^*(e^{-j\omega}) \Rightarrow |X(e^{j\omega})| = |X^*(e^{-j\omega})|$$

↗ Fazlar orjine göre simetriktir

∗ Reel bir işaretin fourier dönüşümü ω'ya simetriktir

orj. s. 4

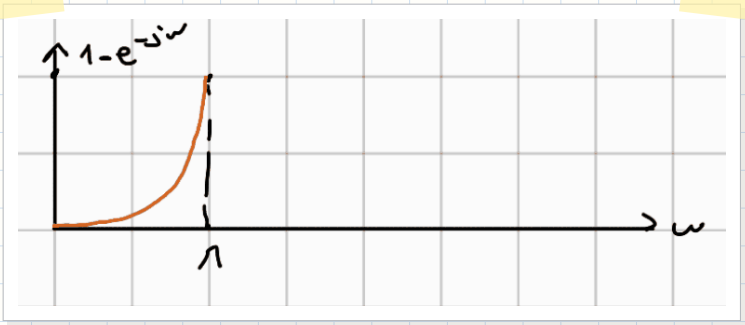
$$x[n] \longleftrightarrow X(e^{j\omega})$$
$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

ispat

$$\begin{cases} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\ X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[-n]\, e^{-j\omega n} \\ X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[k]\, e^{j\omega k} \\ \qquad\qquad\qquad\qquad \downarrow\ e^{-j(-\omega)k} \end{cases}$$

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) - e^{-j\omega} X(e^{j\omega})$$
$$= (1 - e^{-j\omega})\, X(e^{j\omega})$$

? türev alabilir



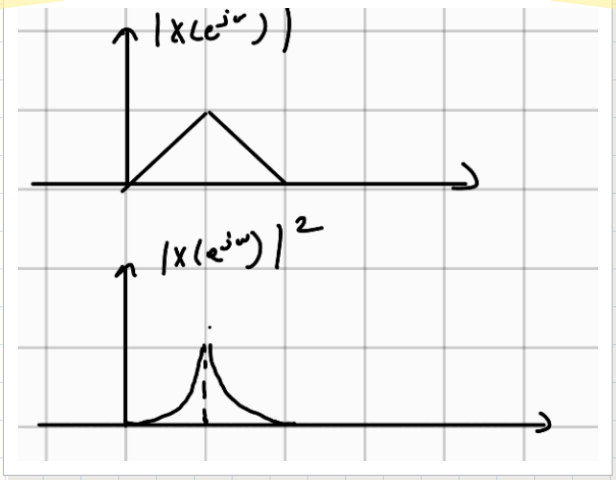
orijinal çizim, s. 4

yüksek frekanslar güçlenir

$$n\,x[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega$$

x[n] işaretinin enerjisi Enerji yoğunluğu



orijinal çizim, s. 4

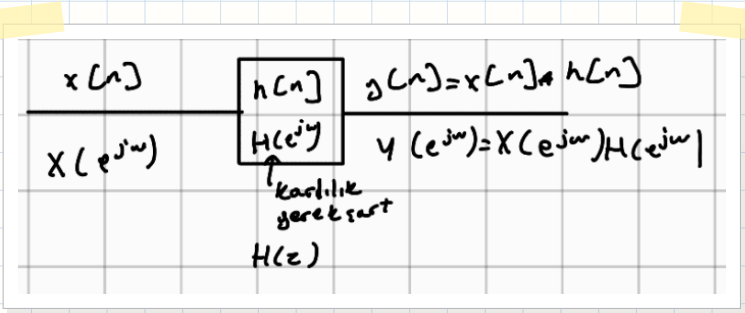
The convolution Theorem

① $x_1[n] * x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})\,X_2(e^{j\omega})$

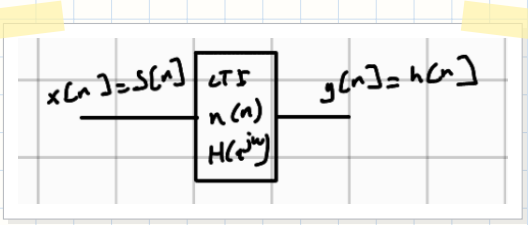
② $x_1[n]\,x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(e^{j\omega})\,X_2(e^{j\omega})$[4]

$y[n] = x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

[4] Claude notu: muhtemelen $X_1(e^{j\omega})$ ile $X_2(e^{j\omega})$ arasında periyodik konvolüsyon ($\circledast$) olmalı.

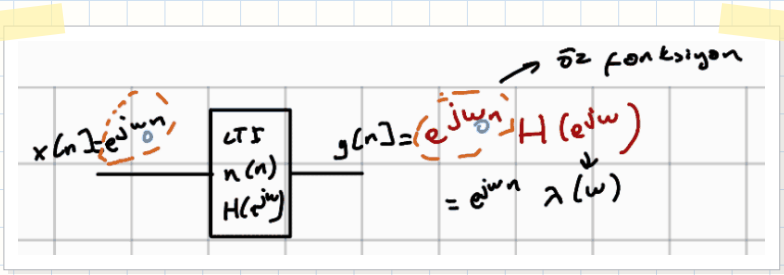


orijinal çizim, s. 4



orijinal çizim, s. 4

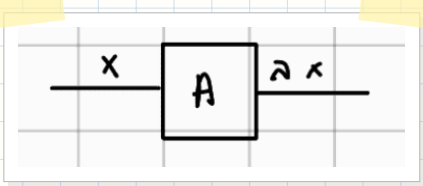
orj. s. 5



orijinal çizim, s. 5

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\,e^{j\omega(n-k)} \\
&= \left[\sum_{k=-\infty}^{\infty} h[k]\,e^{-j\omega k}\right] e^{j\omega n} \\
&= e^{j\omega n}\,H(e^{j\omega})
\end{aligned}
$$

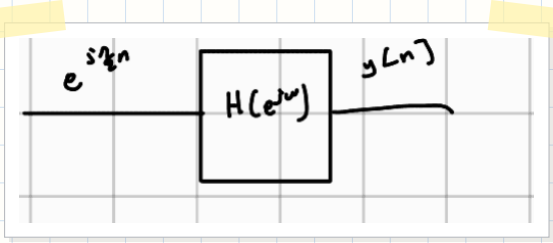
özvektör



orijinal çizim, s. 5

$$A\vec{x} = \lambda\vec{x}$$

eigen value (öz değer) eigen vector (öz vektör)



orijinal çizim, s. 5

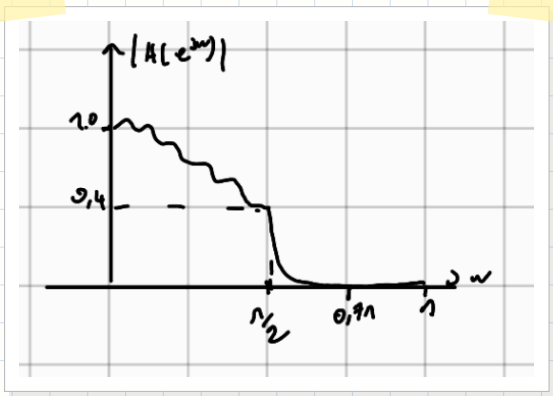
$$y[n] = e^{j\frac{\pi}{2}n} H(e^{j\frac{\pi}{2}})$$

dizi katsayı (örnek $0{,}4\,e^{j\theta}$)

$\omega = \frac{\pi}{2}$ giriş için

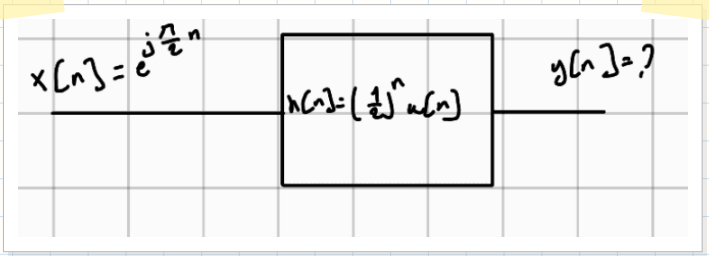
frekans değişmiyor

Lineer bir sistem giriş işaretin frekansı korunur.
Fakat 0'a çevirebilir.



orijinal çizim, s. 5

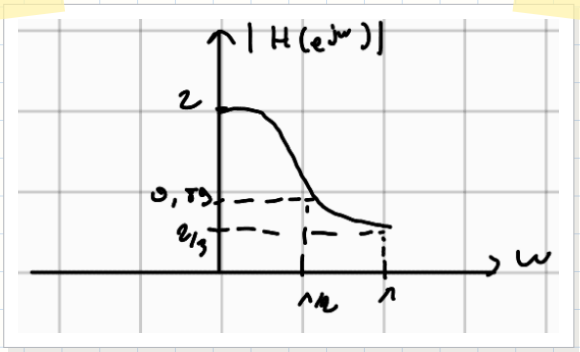
Girişin frekansına göre sistem farklı davranır bu yüzden $H(e^{j\omega})$'ya frekans cevabı denir.



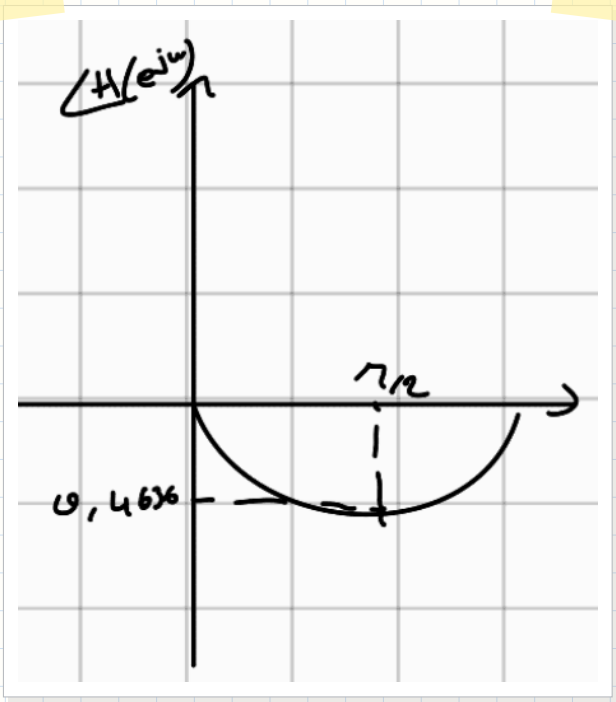
orijinal çizim, s. 5

orj. s. 6

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n]\, e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} \underbrace{\left(\tfrac{1}{2} e^{-j\omega}\right)}_{\alpha}{}^n = \frac{1}{1 - \tfrac{1}{2} e^{-j\omega}}$$



orijinal çizim, s. 6



orijinal çizim, s. 6

$$y[n] = e^{j\frac{\pi}{2}n} \underbrace{H(e^{j\frac{\pi}{2}})}_{0{,}89\, e^{-j\,0{,}4636}}$$
$$= 0{,}89\, e^{j(\frac{\pi}{2}n - 0{,}4636)}$$