Sayısal İşaret İşleme

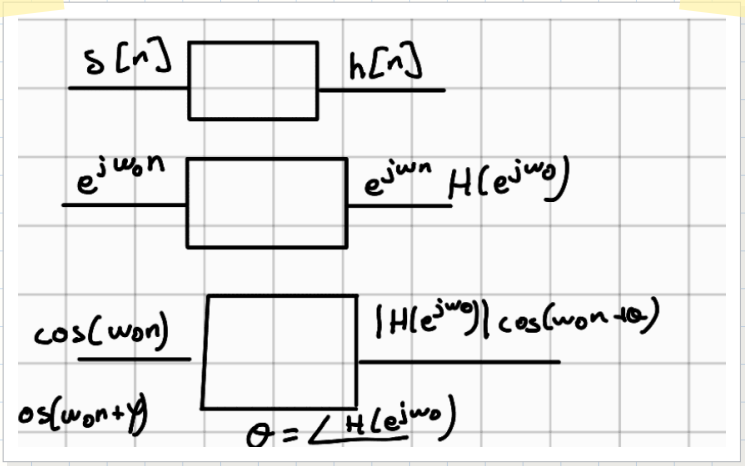
Frekans yanıtı, DTFT ve Z dönüşümü

Orijinal not: Dsp-ders-4.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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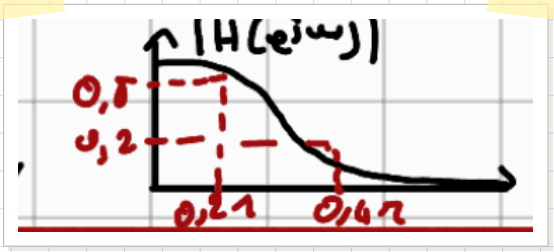
orj. s. 1



orijinal çizim, s. 1

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\theta}$$
$$\theta = \angle H(e^{j\omega})$$

$$\cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$
$$y[n] = \frac{1}{2} e^{j\omega_0 n} H(e^{j\omega_0}) + \frac{1}{2} e^{-j\omega_0 n} H(e^{-j\omega_0})$$



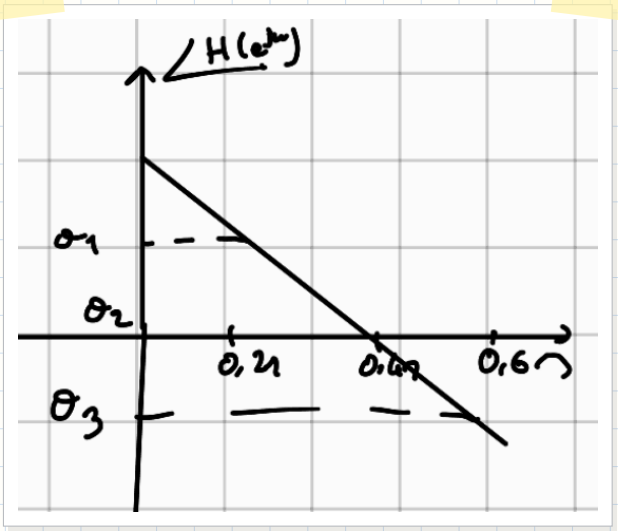
orijinal çizim, s. 1

$$h[n] \text{ reel} \Rightarrow H(e^{j\omega}) = H^*(e^{-j\omega})$$
$$H(e^{-j\omega}) = H^*(e^{j\omega})$$

← **önemli**

$$h[n] \text{ reel} \Rightarrow y[n] = \tfrac{1}{2}e^{j\omega_0 n}H(e^{j\omega_0}) + \tfrac{1}{2}e^{-j\omega_0 n}H^*(e^{j\omega_0})$$
$$= \tfrac{1}{2}e^{j\omega_0 n}|H(e^{j\omega_0})|\,e^{j\theta_0} + \tfrac{1}{2}e^{-j\omega_0 n}|H(e^{j\omega_0})|\,e^{-j\theta_0}$$
$$= |H(e^{j\omega_0})|\left[\frac{e^{j(\omega_0 n+\theta_0)} + e^{-j(\omega_0 n+\theta_0)}}{2}\right]$$
$$= |H(e^{j\omega_0})|\cos(\omega_0 n + \theta_0)$$

$$x[n] = \cos(0{,}1\pi n) + \cos(0{,}4\pi n) + \cos(0{,}6\pi n)$$
$$y[n] = 0{,}8\cos(0{,}1\pi n + \theta_1) + 0{,}2\cos(0{,}4\pi n + \theta_2) + 0\cdot\cos(0{,}6\pi n + \theta_3)$$



orijinal çizim, s. 1

$$|H(e^{j\frac{\pi}{2}})|\cos\left(\tfrac{\pi}{2}n + \angle H(e^{j\frac{\pi}{2}})\right)$$

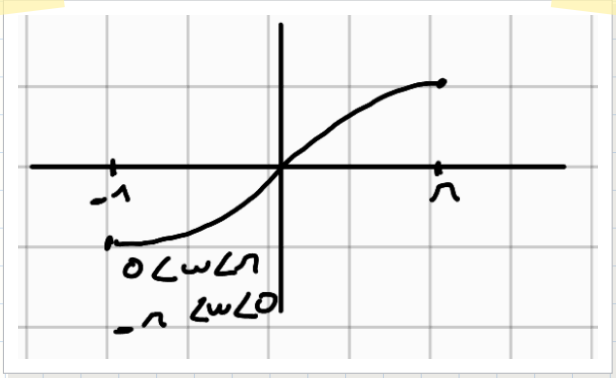
$$x[n] = \cos\left(\tfrac{\pi}{2}n\right)$$
$$h[n] = \delta[n] - \delta[n-1]$$

⚠ **Dikkat!**
Sakın ha $y[n] = \cos\left(\tfrac{\pi}{2}n\right) - \cos\left(\tfrac{\pi}{2}(n-1)\right)$

$$y[n] = ?$$

$$H(e^{j\omega}) = 1 - e^{-j\omega} = 2j\,\frac{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}{2j}$$
$$= j\,e^{-j\omega/2}\,2\sin(\omega/2)$$
$$|H(e^{j\omega})| = 2\,|\sin(\omega/2)|$$

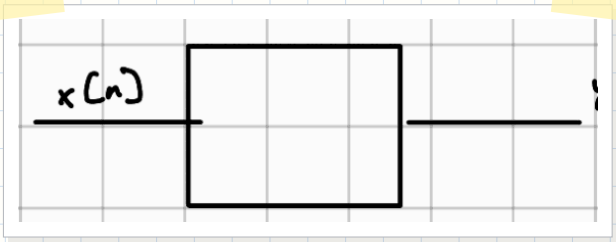
$$\angle H(e^{j\omega}) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2}, & 0 < \omega < \pi \\ \frac{\pi}{2} - \frac{\omega}{2} + \pi, & -\pi < \omega < 0 \end{cases}$$



orijinal çizim, s. 1

orj. s. 2

$$y[n] = \sqrt{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)$$



orijinal çizim, s. 2

$$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$$

$$h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

$$H(e^{j\omega}) = \frac{1}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}$$
$$= \sum_{k=0}^{2} \frac{1}{3}e^{-j\omega k}$$

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$ ← önemli

$$= \frac{1}{3}\frac{(e^{-j\omega})^0 - (e^{-j\omega})^{2+1}}{1-e^{-j\omega}}$$

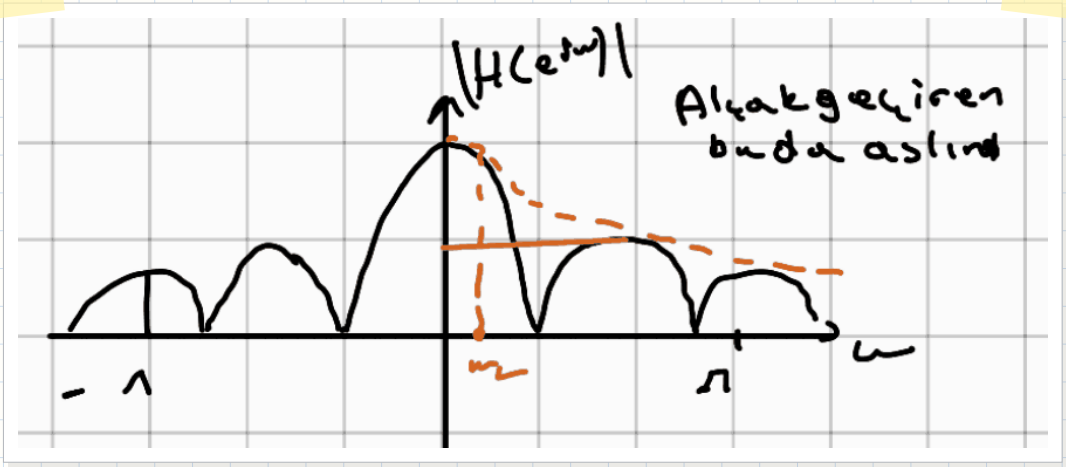
$$= \frac{1}{3}\frac{1-e^{-j3\omega}}{1-e^{-j\omega}} \quad \text{ikisini eşit bölse} \quad \frac{1}{3}\frac{e^{-j\frac{3}{2}\omega}\left(e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= \frac{1}{3}e^{-j\omega}\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}$$

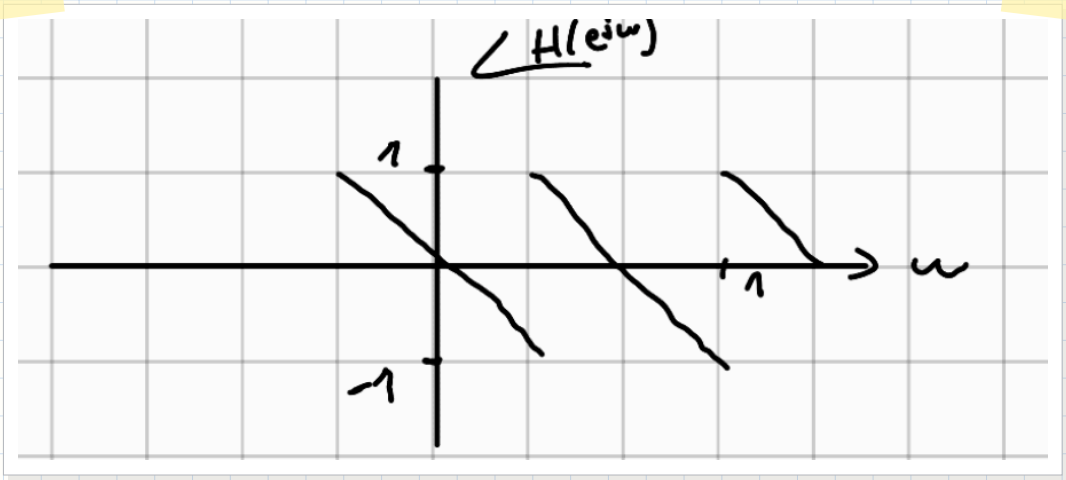
> ⚠ **Dikkat!**
> Her zaman sin hale getir

$$|H(e^{j\omega})| = \frac{1}{3}\left|\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right|$$

$$\angle H(e^{j\omega}) = -\omega + \pi \quad \text{(negatif olduğunda)}$$

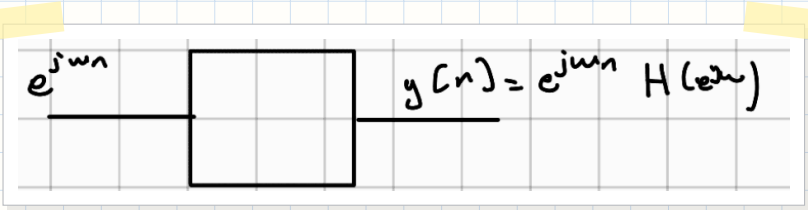


orijinal çizim, s. 2

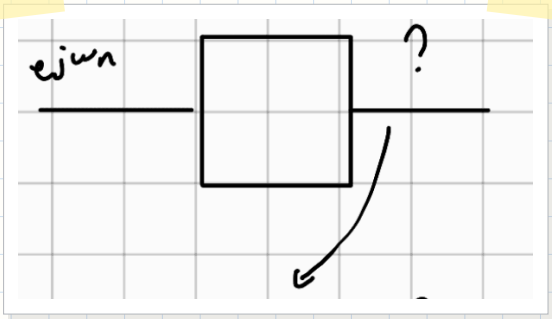


orijinal çizim, s. 2

orj. s. 3

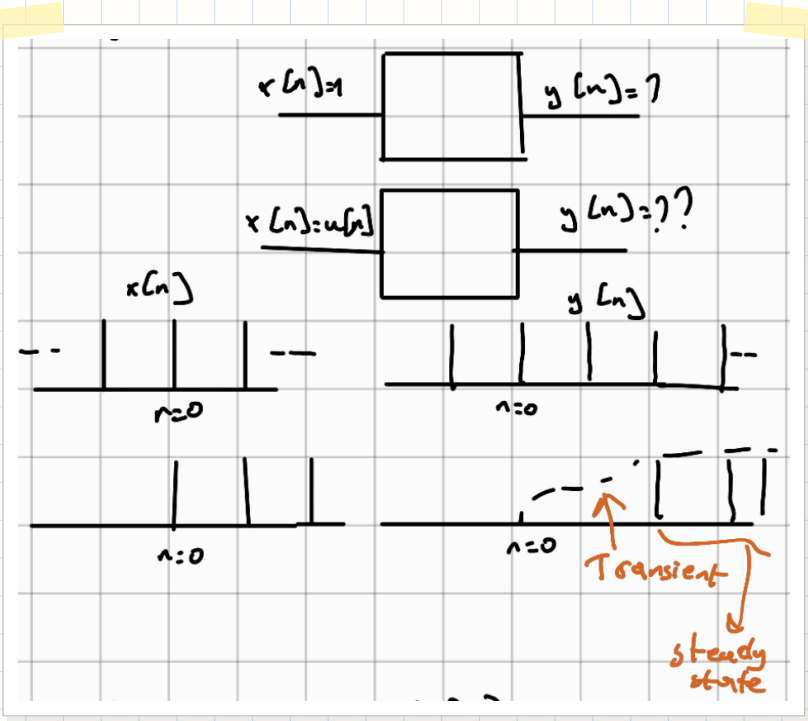


orijinal çizim, s. 3



orijinal çizim, s. 3

$$y[n] = \begin{cases} 0, & n < 0 \\ ?, & n \geq 0 \end{cases}$$



orijinal çizim, s. 3

$$y[n] = \sum_k h[k]\,x[n-k]$$

system is causal $\Rightarrow h[n] = 0, \quad n < 0$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)}\, u[n-k]$$

$$= \sum_{k=0}^{\infty} h[k]\, e^{j\omega(n-k)}\, u[n-k]$$

$h[n]$ casual

$$u[n-k] = 0, \quad \begin{matrix} n-k<0 \\ k>n \end{matrix}$$

← önemli

$$y[n] = \sum_{k=0}^{n} h[k]\, e^{j\omega(n-k)}$$

$$= \sum_{k=0}^{\infty} h[k]\, e^{-j\omega k}\, e^{j\omega n} - \sum_{k=n+1}^{\infty} h[k]\, e^{-j\omega k}\, e^{j\omega n}$$

$$= \underbrace{e^{j\omega n} H(e^{j\omega})}_{\text{steady state cevabı}} - \sum_{k=n+1}^{\infty} h[k]\, e^{-j\omega k}\, e^{j\omega n}$$

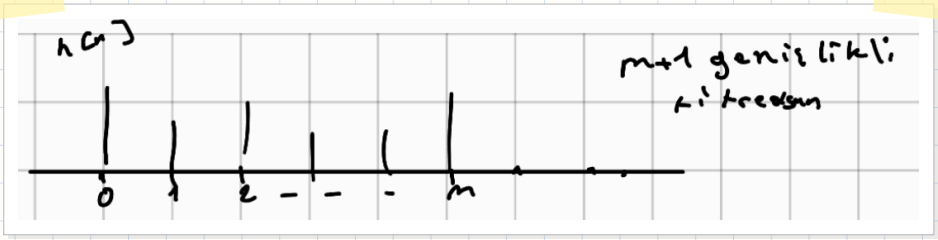
$$y[n] = y_{ss}[n] + y_{tr}[n]$$

$$y[n] = y_{ss}[n] + y_{tr}[n]$$

$$y_{ss}[n] = e^{j\omega n} H(e^{j\omega})$$

$$y_{tr}[n] = -\sum_{k=n+1}^{\infty} h[k]\, e^{-j\omega k}\, e^{j\omega n}$$

orj. s. 4



orijinal çizim, s. 4

Filtrenin genişliği $m+1$

$$y_{tr}[0] = -\sum_{k=1}^{\infty}(\quad) \neq 0$$

$$y_{tr}[1] = -\sum_{k=2}^{\infty}(-\ -) \neq 0$$

$$y_{tr}[2] = -\sum_{k=3}^{\infty}(-\ -\ -) \neq 0$$

$$\vdots$$

$$y_{tr}[m-1] = -\sum_{k=m}^{\infty}(-\ -\ -) \neq 0$$

$$y_{tr}[m] = \sum_{k=m+1}^{\infty}(-\ -\ -) = 0$$

$$y_{tr}[m+1] = \sum_{k=m+2}^{\infty}(-\ -\ -) = 0$$

pencere genişliği -1'e kadar 0'dan farklı;

pencere genişliği ne kadar dar olursa o kadar pencere genişliği dar olur

$$h[n] = \tfrac{1}{3}\delta[n] + \tfrac{1}{3}\delta[n-1] + \tfrac{1}{3}\delta[n-2]$$

$x[n] = 1$ olsaydı, $y[n] = 1$ olurdu

$$x[n] = 1 \cdot u[n]$$

$$y[n] = 0, \quad n < 0$$

$$\left.\begin{aligned} y[0] &= \tfrac{1}{3} \\ y[1] &= \tfrac{2}{3} \end{aligned}\right\} \text{Transient iki örnek } \boxed{?\ \text{sürdü}}$$

$$y[2] = 1$$

$$y[3] = 1$$

> ⚠ **Dikkat!**
> pencere genişliği eksi 1'i kadar

Peki transient kararlı mı ??

$$|y_{tr}[n]| = \left|\sum_{k=n+1}^{\infty} h[k]\, e^{-j\omega k}\, e^{j\omega n}\right|$$

$$\leq \sum_{k=n+1}^{\infty} |h[k]| \left|e^{-j\omega k}\, e^{j\omega n}\right|$$

$$\leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=0}^{\infty} |h[k]|$$

orj. s. 5

Sistem kararlı ise transient'i de kararlıdır.

3. bölüm

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

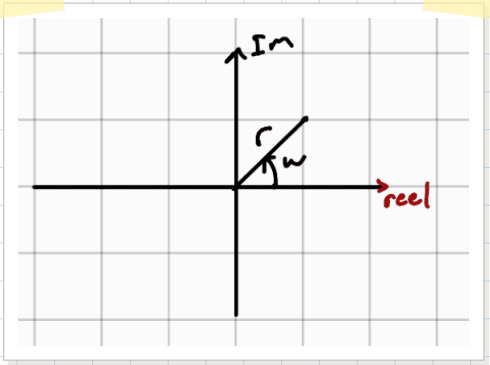
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$z = re^{j\omega}$$

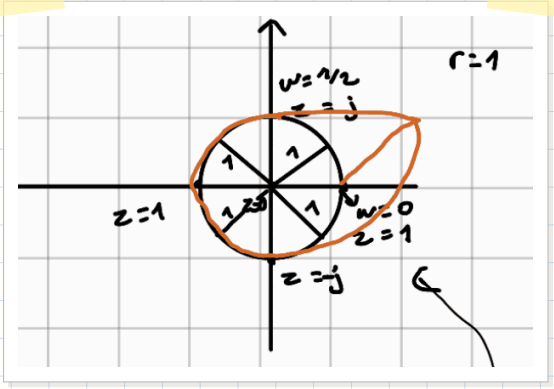
$$= \sum_{n=-\infty}^{\infty} x[n]\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[x[n]\, r^{-n}\right] e^{-j\omega n}$$

z dönüşümü $x[n]r^{-n}$'in Fourier Transform
kararlılık analizi için kullanılır
$z = re^{j\omega}$

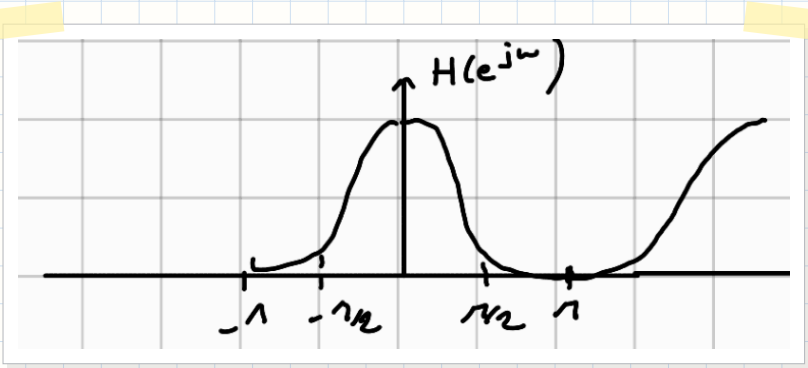


orijinal çizim, s. 5



orijinal çizim, s. 5

$r = 1$ için z transformu ile ?Fourier dönüşümü aynıdır



orijinal çizim, s. 5

Sonuç, birim çemberin üzerindeyse sistem kararlıdır

$$x[n] \longleftrightarrow X(z)$$
$$h[n] \longleftrightarrow H(z)$$
$$y[n] \longleftrightarrow Y(z)$$

$$X(z) = \frac{P(z)}{Q(z)} \quad \text{(rasyonel)}$$

$P(z)$'nin kökleri $X(z)$ sıfırı
$Q(z)$'nin kökleri kutup

orj. s. 6

Örnek 1

$x[n] = a^n u[n], \quad X(z) = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] \cdot z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} \left(\frac{a}{z}\right)^n u[n]$$
$$= \frac{1}{1-\frac{a}{z}} \qquad \left|\frac{a}{z}\right| < 1 \qquad |z| > |a|$$

$\sum_{n=0}^{\infty} a^n$

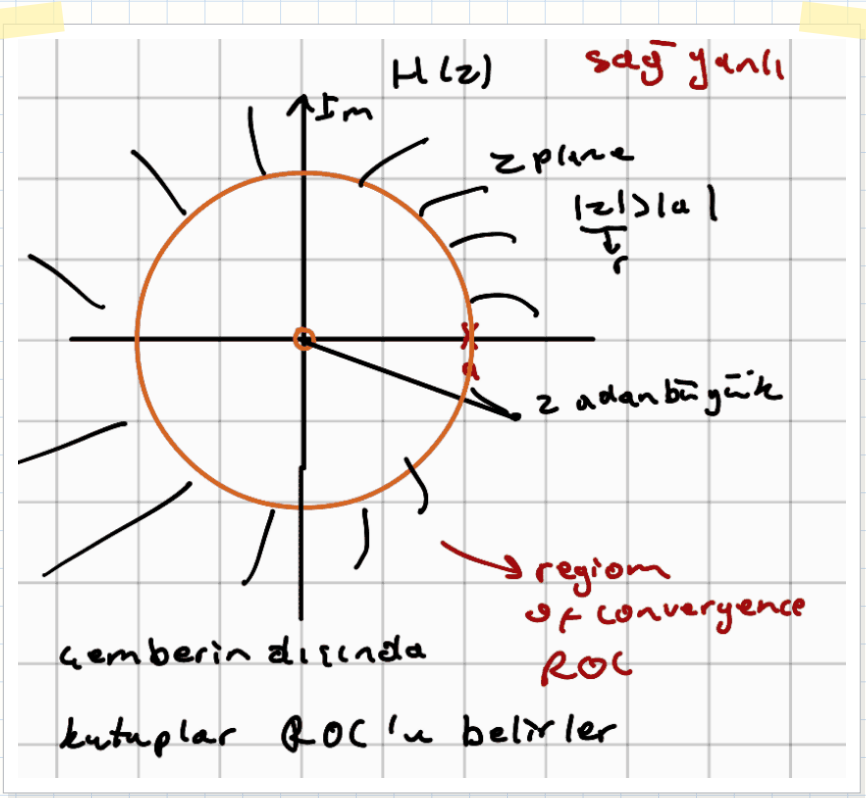
$$X(z) = \frac{1}{1-az^{-1}}$$

$$X(z) = \frac{1}{1-az^{-1}} \rightarrow \text{negatif form}$$
$$= \frac{z}{z-a}$$

⚠ Dikkat!

? mutlaka bu forma getir.

sıfırda bir sıfırı var ? baştan göremedik



orijinal çizim, s. 6

Örnek 2

$x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty} -\left(\frac{a}{z}\right)^{-n} = \sum_{n=0}^{\infty} -\left(\frac{a}{z}\right)^{-n} - [-1]$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1$$

← önemli

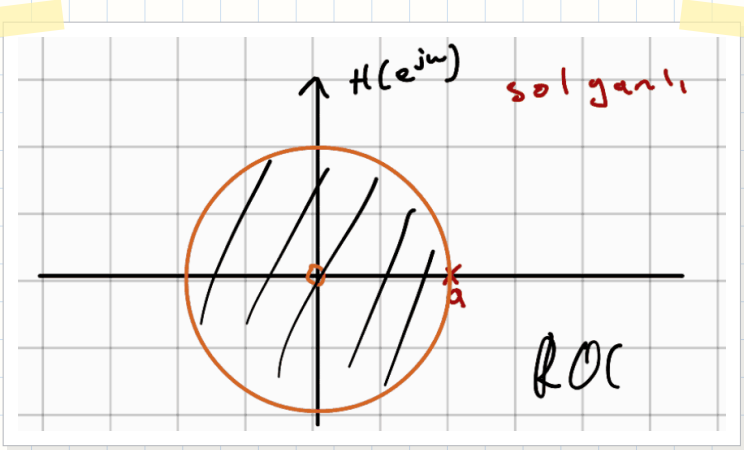
$$= 1 - \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$

$$= 1 - \frac{1}{1-\frac{z}{a}} \qquad |a| < 1 \quad \left|\frac{z}{a}\right| < 1$$

$$= 1 - \frac{a}{a-z} = \frac{-z}{a-z}$$

$$= \frac{z}{z-a} \qquad \left|\frac{z}{a}\right| < 1 \quad |z| < a$$

z dönüşümü ROC'la birlikte anlamlıdır



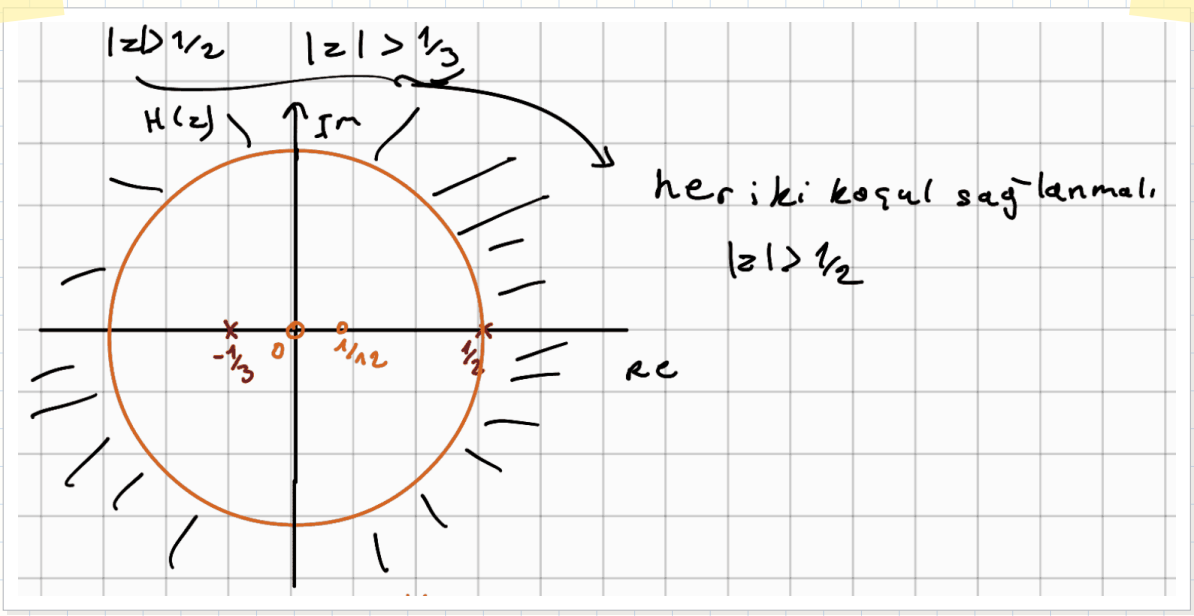
orijinal çizim, s. 6

orj. s. 7

Örnek 3

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad |z| > \frac{1}{2}$$

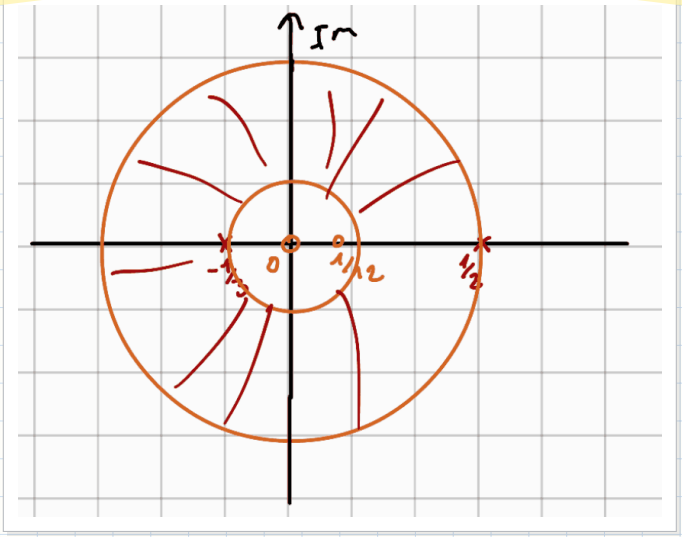


orijinal çizim, s. 7

Örnek 4

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \underbrace{\frac{z}{z+\frac{1}{3}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{z}{z-\frac{1}{2}}}_{|z|<\frac{1}{2}} = \underbrace{\frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}}_{\frac{1}{3}<|z|<\frac{1}{2}}$$



orijinal çizim, s. 7

orj. s. 8

Örnek 5

$$x[n] = \left(-\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[-n-1]$$

$$X(z) = \underset{|z|>\frac{1}{2}}{\frac{z}{z+\frac{1}{2}}} + \underset{|z|<\frac{1}{3}}{\frac{z}{z-\frac{1}{3}}}$$

ROC yok
hiç bir zaman çakışmaz

Örnek 6

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{o.w.} \end{cases}$$

$$X(z) = \sum_{0}^{N-1} a^n z^{-n} = 1 + az^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-(N-1)}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots + \frac{a^{N-1}}{z^{N-1}}$$

$z = 0$'da toplam sonsuz

z hariç her yerde z dönüşümü vardır

$$\frac{1-a^N}{1-az^{-1}}^{1}$$

Örnek 7

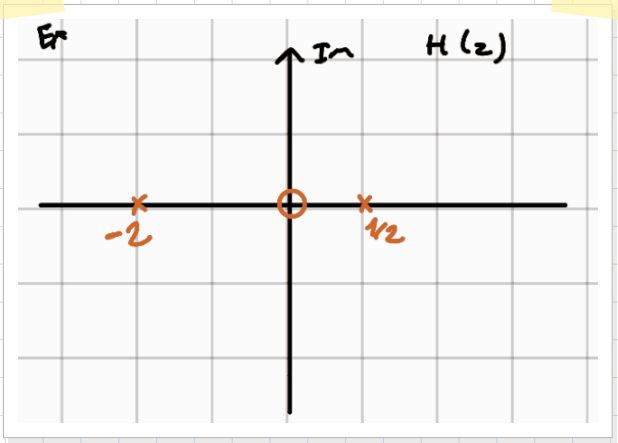
$$x[n] = \begin{cases} a^n, & -N \le n \le 0 \\ 0, & \text{o.w} \end{cases}$$

$$X(z) = 1 + a^{-1}z + a^{-2}z^2 + \cdots + a^{-N}z^N$$

[1] Claude notu: muhtemelen pay $1 - a^N z^{-N}$ olmalı.

$z = \infty$ için ∞ olur

$z \neq \infty$ için z dönüşümü vardır



orijinal çizim, s. 8

$$H(z) = K \frac{z}{\left(z - \frac{1}{2}\right)(z + 2)}$$

? Köksal'ın K'sı :)

$|z| > 2$ ✓ → causal

$\frac{1}{2} < |z| < 2$ ✓ stable.

$|z| < \frac{1}{2}$ ne nedensel ne kararlıdır

kararlı ise ROC birim çemberi içermeli

⋆ Sorunun ? içinde nedensel derse sağ yanlıdır