Sayısal İşaret İşleme

Z dönüşümünün özellikleri

Orijinal not: Dsp-ders-5.pdf

UYARI Bu belge, el yazısı ders notlarından Claude (Anthropic) yardımıyla dijitale aktarılmıştır. Yardımcı kaynak olarak hazırlanmıştır; hata veya eksik içerebilir. Lütfen orijinal notları ve ders kaynaklarını esas alın.

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Not: Orijinal dosyadaki basılı sayfalar (s. 2) dijitalleştirilmedi; yalnızca el yazısı sayfalar aktarıldı.

orj. s. 1

Ders 9

Z dönüşümünün özellikleri

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \qquad \text{ROC: } R_x$$
$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \qquad \text{ROC: } R_{x_1}$$
$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z) \qquad \text{ROC: } R_{x_2}$$

$$a\,x[n] \overset{\mathcal{Z}}{\longleftrightarrow} a\,X(z) \qquad \text{ROC: } R_x$$
$$a_1x_1[n] + a_2x_2[n] \longleftrightarrow a_1X_1(z) + a_2X_2(z) \qquad \text{ROC: } R_{x_1} \cap R_{x_2}$$
$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z) \qquad \text{ROC: } R_x$$
$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0) \qquad \text{ROC: } |z_0|R_x$$

Örnek

$$\underbrace{e^{j\omega_0 n}}_{z_0}\, x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

$$z_0^n x[n] \longleftrightarrow X\left(z/e^{j\omega_0}\right) \qquad \text{ROC: } R_x$$

$$z_0 = e^{j\omega_0}$$
$$|e^{j\omega_0}| = 1 \qquad X\left(e^{-j\omega_0}z\right)$$
$$\angle e^{j\omega_0} = \omega_0$$

Örnek

$$x[n] = r^n \cos(\omega_0 n)\, u[n]$$

$$= r^n \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right) u[n]$$

$$= \frac{1}{2} \left(\underbrace{re^{j\omega_0}}_{z_1} \right)^n u[n] + \frac{1}{2} \left(\underbrace{re^{-j\omega_0}}_{z_2} \right)^n u[n]$$

$$\Rightarrow X(z) = \frac{1}{2} \frac{1}{1 - z^{-1}} \Big|_{z \leftarrow z/z_1} + \frac{1}{2} \frac{1}{1 - z^{-1}} \Big|_{z \leftarrow z/z_2}$$

$$\text{ROC: } |z| > r$$

? okunamadı

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| > 1 \rightarrow \text{ROC: } |z| > 1 \cdot r$$

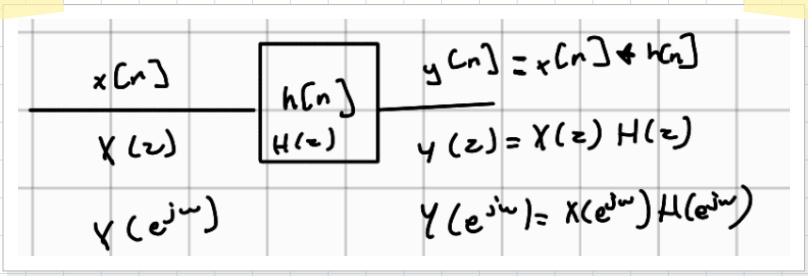
$$n x[n] \longleftrightarrow -z \frac{dX(z)}{dz} \qquad \text{ROC: } R_x$$

$u[n]$	$\cos(\omega_0 n)$
$n\,u[n]$	$n\cos(\omega_0 n)$

orj. s. 3

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) X_2(z)$$

$$\text{ROC: } R_{x_1} \cap R_{x_2}$$



orijinal çizim, s. 3

constant coeff. diff equation

$H(z) \rightarrow$ sistem fonksiyonu

$H(s) \rightarrow$ Transfer fonksiyonu

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) \longleftrightarrow x[n] = ??$$

$$u[n] \longleftrightarrow \frac{1}{1 - z^{-1}}$$

1) Tabloya bakma yöntemi:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}$$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3} \rightarrow x[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$X(z) = \frac{0{,}5z}{z^2 - z + 0{,}25} \qquad |z| > 0{,}5$$

$$X(z) = \frac{0{,}5z}{(z-0{,}5)^2} = \frac{0{,}5z^{-1}}{(1-0{,}5z^{-1})^2}$$

$$x[n] = n\left(\frac{1}{2}\right)^n u[n]$$

⚠ Dikkat!
z dönüşümünün ? türevi bulunur

orj. s. 4

2)

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad \frac{1 + 2z^{-1} + 3z^{-5}}{2 + 3z^{-4}}$$

$$= \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} \qquad \frac{(1-3z^{-1})(1-2z^{-1})}{(1-z^{-1})(1-4z^{-1})}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \quad (M < N) \qquad \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) + \left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) + \cdots$$

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z \leftarrow d_k}$$

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \qquad |z| > \frac{1}{2}$$

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

? $A_1 \leftarrow -1$, $A_2 \leftarrow 2$

$$A_1 = \left(1-\frac{1}{4}z^{-1}\right) X(z)\Big|_{z \leftarrow \frac{1}{4}} = \frac{1}{1-\frac{1}{2}z^{-1}}\Big|_{z \leftarrow \frac{1}{4}} = \frac{1}{1-2} = -1$$

$$A_2 = \frac{1}{1-\frac{1}{4}z^{-1}}\Big|_{z \leftarrow \frac{1}{2}} = 2$$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

$|z| < \frac{1}{2}$ olsaydı,

$$-\left(\frac{1}{2}\right)^n u[-n-1] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}$$

$$+\left(\frac{1}{4}\right)^n u[-n-1]$$

$$-\left(\frac{1}{2}\right)^n u[-n-1]$$

→ _____

orj. s. 5

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \qquad (M < N)$$

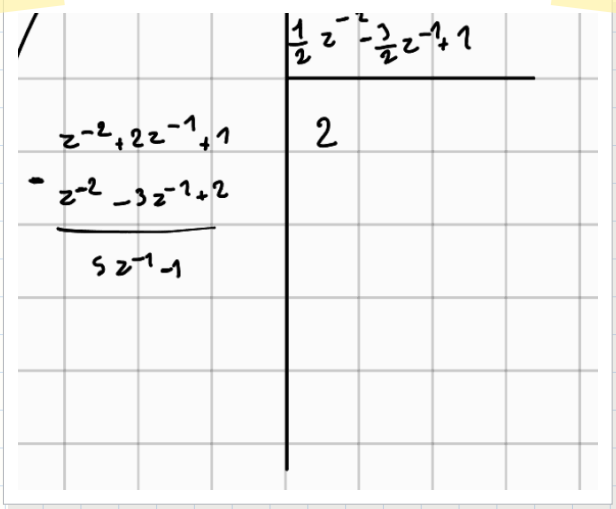
$$= \underbrace{\sum_{r=0}^{M-N} B_r z^{-r}}_{B_0+B_1 z^{-1}+B_2 z^{-2}} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \qquad (M \geq N)$$

$$\frac{1+2z^{-1}+3z^{-2}}{1+z^{-1}} = \frac{z^2+2z+3}{z^2+z} \qquad \begin{matrix} M=2 \\ N=1 \end{matrix}$$

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} \qquad |z| > 1$$

$M = 2 \quad N = 2 \quad M = N$

$$= \boxed{2} + \underbrace{\frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}}_{M<N}$$



orijinal çizim, s. 5

$$2 + \frac{-1 + 5z^{-1}}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)} = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

$$\Rightarrow x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

orj. s. 6

Çok katlı kök için

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

(s katlı)

$$C_m = \frac{1}{(s-m)!\,(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[(1 - d_i w)^s X(w^{-1}) \right] \right\}_{w = d_i^{-1}}$$

$z^{-1} = w$

$$\frac{1}{(2-1)!\,(-d_i)^{2-1}} \left\{ \frac{d}{dw} \left[(1 - d_i w)^2 X(w^{-1}) \right] \right\}_{w = d_i^{-1}}$$

$$X(z) = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right)} \qquad |z| > \frac{1}{2}$$

$\frac{1}{2}$'de ikili kutup

$$= \frac{A_1}{1 - \frac{1}{3}z^{-1}} + \frac{C_1}{1 - \frac{1}{2}z^{-1}} + \frac{C_2}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

$s = 2 \quad m = 1$

$$C_1 = \frac{1}{1!\left(-\frac{1}{2}\right)^1} \left\{ \frac{d}{dw} \left[\left(1 - \frac{1}{2}w\right)^2 \cdot \frac{-11 + 16w - 7w^2}{\left(1 - \frac{1}{2}w\right)^2 \left(1 - \frac{1}{3}w\right)} \right] \right\}_{w=2}$$

$$= 114$$

$s = 2 \quad m = 2$

$$C_2 = \left. \frac{-11 + 16w - 7w^2}{1 - \frac{1}{3}w} \right|_{w=2} = -21$$

$$\rightarrow x[n] = -104\left(\frac{1}{3}\right)^n u[n] + 114\left(\frac{1}{2}\right)^n u[n] - 21(n+1)\left(\frac{1}{2}\right)^{n+1} u[n+1]$$

Tabloda bu var

$$\left[\frac{-21z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} \right] \text{ z ile çarpınız} \quad = \quad \frac{-21}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

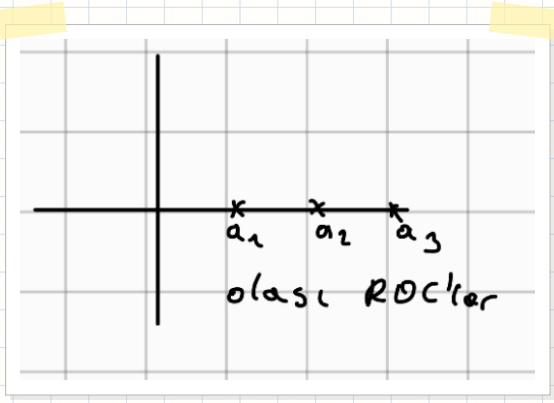
$$a_{-3} = 0$$
$$x[-2] = a_{-2} = 1$$
$$x[-1] = a_{-1} = -\frac{1}{2}$$
$$x[0] = a_0 = -1$$
$$x[1] = a_1 = \frac{1}{2}$$
$$x[2] = a_2 = 0$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

$$x = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

orj. s. 7

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)} \qquad \begin{matrix} |z| > 1 \\ |z| < \frac{1}{2} \end{matrix}$$



orijinal çizim, s. 7

sağ yanlı dizi için $n > 0$

$|z| > 1$? durumunda

$$1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \cdots$$

sol yanlı

$|z| < \frac{1}{2}$

$$= 2z^2 + 6z^3 + 14z^4 + 30z^5 + \cdots$$

Peki nasıl bulacağız